REVIEW OF THE CHAPTER 12

Problem 1. Find the total differential of the function $f(x,y)=xye^{\frac{x}{y}}$ at the point $(0,1)$.

Solution. The total differential of the function $z=f(x,y)$ given by formula

$$dz=f_x(x,y)dx+f_y(x,y)dy.$$

For our function we have

$$f_x(x,y)=ye^{\frac{x}{y}}+xe^{\frac{x}{y}},\quad f_y(x,y)=xe^{\frac{x}{y}}-\frac{x^2}{y}e^{\frac{x}{y}}$$

Next we need to find the values of the partial derivatives at the point $(0,1)$.

$$f_x(0,1)=1\quad f_y(0,1)=0.$$

Hence

$$dz=dx.$$

Problem 2. The surface is given by equation $F(x,y,z)=x^2+y^2-z^2=0$. Find equations of a normal line and a tangent plane to this surface at the point $P=(1,0,1)$.

Solution. The equation of the tangent plane to the surface $F(x,y,z)=0$ at the point $P=(x_0,y_0,z_0)$ is given by formula

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0. \quad (1)$$

And the equation of the normal line to the surface $F(x,y,z)=0$ at the point $P=(x_0,y_0,z_0)$ is given by formula

$$\left\{\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} x_0\\ y_0\\ z_0\end{pmatrix}+\begin{pmatrix} F_x(x_0,y_0,z_0)\\ F_y(x_0,y_0,z_0)\\ F_z(x_0,y_0,z_0)\end{pmatrix}t.\right. \quad (2)$$

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Let us find the partial derivatives of the function F

$$F_x(x,y,z) = 2x \quad F_y(x,y,z) = 2y \quad F_z(x,y,z) = -2z.$$

So

$$F_x(1,0,1) = 2, \quad F_y(1,0,1) = 0, \quad F_z(1,0,1) = -2.$$

Therefore the equation of the tangent line is

$$2(x-1) - 2(z-1) = 0.$$

The equation of the normal line is

$$x = 1 + 2t \quad y = 0 \quad z = 1 - 2t.$$

Problem 3. Find points the local maximum, local minimum points and the saddle point of the function $f(x,y) = x^3 - 6xy + 8y^3$.

Solution. First we find a critical points of this function. We remind that a point (x,y) is called the critical point if

$$\nabla f(x,y) = 0.$$

The partial derivatives of the function f are given by formulas

$$f_x(x,y) = 3x^2 - 6y, \quad f_y(x,y) = -6x + 24y^2.$$

Solving the system

$$\begin{cases} x^2 - 2y = 0 \\ -x + 4y^2 = 0 \end{cases}$$

we have

$$16y^4 - 2y = 0.$$

This equation has two solutions

$$y = 0 \quad \text{and} \quad y = \frac{1}{2}.$$

If $y = 0$ then $x = 0$. If $y = \frac{1}{2}$ then $x = 1$. Now we need to use the second derivatives test.

Theorem 2. Second Derivatives Test. *Suppose the second partial derivatives of the function f are continuous in a disk with center (a, b) and suppose that $\nabla f(a, b) = 0$. Let*

$$D = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2$$

Then

$$a) If \ \ D > 0 \ \ and \ \ f_{xx}(a, b) > 0 \ \ then \ \ (a, b) \ \ is \ \ a \ \ point \ \ of \ \ local\,minimum.$$

$$b) If \ \ D > 0 \ \ and \ \ f_{xx}(a, b) < 0 \ \ then \ \ (a, b) \ \ is \ \ a \ \ point \ \ of \ \ local\,maximum.$$

$$c) If \ \ D < 0 \ \ then \ \ (a, b) \ \ is \ \ not \ \ local \ maximum \ or \ minimum.$$

The second order partial derivatives of the function $f(x, y)$ are

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 48y, \quad f_{xy}(x, y) = -6.$$

Now we compute the function $\mathcal{D}$

$$\mathcal{D}(0, 0) = f_{xx}(x, y) f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 288xy - 6^2.$$

Since $D(0, 0) < 0$ the point $(0, 0)$ is the saddle point. On the other hand

$$\mathcal{D}(0, \frac{1}{2}) = 148 - 36 = 112 > 0$$

and $f_{xx}(1, \frac{1}{2}) > 0$ so the point $(1, \frac{1}{2})$ is the point of local minimum.

problem 4. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

Solution. The square of the distance between a point $P = (x, y, z)$ and the origin is given by function $f(x, y, z) = x^2 + y^2 + z^2$. So we need to minimize this function under the constrain equation

$$g(x, y, z) = xy^2z^3 - 2.$$

We use the method of Lagrange multipliers. First we the gradients of the functions f and g:

$$\nabla f(x, y) = (2x, 2y, 2z), \quad \nabla g(x, y, z) = (y^2z^3, 2xyz^3, 3xy^2z^2).$$

Therefore we have

$$\begin{cases} 2x + \lambda y^2 z^3 = 0 \\ 2y + 2\lambda xyz^3 = 0 \\ 2z + 3\lambda xy^2 z^2 = 0 \\ xy^2 z^3 - 2 = 0 \end{cases} \tag{3}$$

Obviously $x \neq 0$ $y \neq 0$ $z \neq 0$ and $\lambda \neq 0$. So we can simplify the system (3) as

$$\begin{cases} 2x + \lambda y^2 z^3 = 0 \\ 2 + 2\lambda xz^3 = 0 \\ 2 + 3\lambda xy^2 z = 0 \\ xy^2 z^3 - 2 = 0 \end{cases} \tag{4}$$

Solving the second equation we have

$$\lambda = -\frac{1}{xz^3}.$$

So we can get rid of the parameter λ in the system (4): So we can simplify the system (3) as

$$\begin{cases} 2x - \frac{y^2}{x} = 0 \\ 2 - 3\frac{y^2}{z^2} = 0 \\ xy^2 z^3 - 2 = 0 \end{cases} \tag{5}$$

we can rewrite this system as

$$\begin{cases} 2x^2 = y^2 \\ 2z^2 = 3y^2 \\ xy^2 z^3 - 2 = 0 \end{cases} \tag{6}$$

We can get rid of the variable y in the second and third equation

$$z^2 = 3x^2, \quad x^3 z^3 = 1.$$

From the first equation $z = \sqrt{3}x$. So from the second equation we have

$$3^{\frac{3}{2}} x^6 = 1.$$

Then $x = \frac{1}{3^{\frac{1}{4}}}$ $y = \pm\sqrt{2}\frac{1}{3^{\frac{1}{4}}}$ $z = \sqrt{3}\frac{1}{3^{\frac{1}{4}}} = 3^{\frac{1}{4}}$ and $x = -\frac{1}{3^{\frac{1}{4}}}$ $y = \pm\sqrt{2}\frac{1}{3^{\frac{1}{4}}}$ $z = \sqrt{3}\frac{1}{3^{\frac{1}{4}}} = -3^{\frac{1}{4}}$ If we aguree in some way, we can show that the points $(\frac{1}{3^{\frac{1}{4}}}, \pm\sqrt{2}\frac{1}{3^{\frac{1}{4}}}, 3^{\frac{1}{4}}), (-\frac{1}{3^{\frac{1}{4}}}, \pm\sqrt{2}\frac{1}{3^{\frac{1}{4}}}, -3^{\frac{1}{4}})$ are the points of absolute minimum.