

## STOKES' THEOREM

**Stokes' Theorem.** Let  $S$  be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve  $C$  with positive orientation. Let  $\mathbf{F}$  be a vector field whose components have continuous partial derivatives on an open region in  $\mathbb{R}^3$  that contains  $S$ . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S}. \quad (1)$$

**Example 1.** Evaluate  $\int \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S}$ ,  $\mathbf{F} = xyz\vec{i} + x\vec{j} + e^{xy} \cos(z)\vec{k}$ , and  $S$  is hemisphere  $x^2 + y^2 + z^2 = 1$ ,  $z \geq 0$  oriented upward.

*Solution.* We use the formula (1). Then

$$\int \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

In our case the curve  $C$  is the circle  $x^2 + y^2 = 1$  oriented counterclockwise. Therefore we can parametrize this circle as  $\mathbf{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + 0\vec{k}$ . So we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\cos(t)\sin(t) \cdot 0(-\sin(t)) + \cos(t)\cos(t) + e^{\sin(t)\cos(t)} \cdot 1 \cdot 0) dt = \\ &= \left( \frac{1}{2}t + \frac{1}{4}\sin(2t) \right) \Big|_0^{2\pi} = \pi \end{aligned}$$

**Example 2.** Use Stokes' Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $\mathbf{F} = xz\vec{i} + 2xy\vec{j} + 3xy\vec{k}$ ,  $C$  is the boundary of the part of the plane  $3x + y + z = 3$  in the first octant oriented counterclockwise as viewed from above.

*Solution.* The normal vector to the surface  $S$  is

$$\vec{n} = \frac{1}{\sqrt{11}}(3, 1, 1).$$

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Next we compute  $\text{curl}\mathbf{F}$

$$\text{curl}\mathbf{F} = 3x\vec{i} + (x - 3y)\vec{j} + 2y\vec{k}$$

Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \frac{1}{\sqrt{11}} \int \int_S (10x - 3y + 2y)dS.$$

We note that the surface  $S$  is given as a graph of function  $z = f(x, y) = 3 - 3x - y$ . Therefore  $f_x = -3, f_y = -1$ . Hence

$$\frac{1}{\sqrt{11}} \int \int_S (10x - 3y + 2y)dS = \frac{1}{\sqrt{11}} \int \int_D (10x - 3y + 2y)\sqrt{11}dA$$

Here  $D$  is the triangle

$$D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 3 - 3x\}.$$

Therefore

$$\begin{aligned} \int \int_D (10x - 3y + 2y)dA &= \int_0^1 \int_0^{3-3x} (10x - y)dydx = \int_0^1 (10xy - \frac{y^2}{2})|_0^{3-3x} dx = \\ \int_0^1 (30x - 30x^2 - \frac{9}{2}(1-x)^2)dx &= (15x^2 - 10x^3 - \frac{9}{6}(x-1)^3)|_0^1 = (15 - 10 - \frac{3}{2}) = \frac{7}{2} \end{aligned}$$

**Example 3.** Use Stokes' Theorem to evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $\mathbf{F} = x\vec{i} + y\vec{j} + (x^2 + y^2)\vec{k}$ ,  $C$  is the boundary of the part of the paraboloid  $z = 1 - x^2 - y^2$  in the first octant oriented counterclockwise as viewed from above.

*Solution.*

$$\text{curl}\mathbf{F} = 2y\vec{i} - 2x\vec{j} + 0\vec{k}$$

$$\vec{n} = \frac{1}{\sqrt{1 + 4x^2 + 4y^2}}(2x, 2y, 1)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \frac{1}{\sqrt{11}} \int \int_S 0dS = 0.$$