MAXIMUM/MINIMUM VALUES

Definition 1. *A function of two variables has a local maximum at point* (a, b) *if* $f(x, y) \le f(a, b)$ *for all points* (x, y) *in some disk with center* (a, b)*. The number* $f(a, b)$ *called a local maximum value. If* $f(x, y) \ge f(a, b)$ *for all* (x, y) *in such disk,* $f(a, b)$ *is a local minimum value.*

Definition 2. *A function of two variables has a absolute maximum at point* (a, b) *if* $f(x, y) \le f(a, b)$ *for all points* (x, y) *in domain of function* f*.*

Definition 3. *A function of two variables has a absolute minimum at point* (a, b) *if* $f(x, y) \ge f(a, b)$ *for all points* (x, y) *in domain of function* f*.*

Let a function $f(x, y)$ be a smooth function. We denote the domain of this function as $\mathcal{D}$.

Definition 4. *Let the point* $P = (x_0, y_0)$ *be in* $\mathcal{D}$*. The point* P *is called INTERIOR point if there exists a disk centered at* P *such that this disk is in* $\mathcal{D}$*. The point* P *is called BOUNDARY POINT if each disk centered at* P *has a point which does not belong to* $\mathcal{D}$*.*

Suppose that (a, b) is the point of local maximum or minimum of the function $f(x, y)$. Here and below we assume that the function f has continuous first order derivatives f_x, f_y.

Definition 5. *The point* $P = (x_0, y_0)$ *called the critical point of the function* f *if* $\nabla f(x_0, y_0) = 0$.

We have the following important Theorem.

Theorem 1. *Let the point* (a, b) *be a point of local maximum or minimum of the function* f*. Assume in addition that* (a, b) *is interior point of domain of the function* f*. Then*

$$f_x(a, b) = 0, \quad f_y(a, b) = 0.$$

In other words exach interior point of locla maximum or minimum is critical point.

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Remark. The assumotion that (a, b) is interior point is **Very Important.** Whiothout this assumption in general Theorem 1 is not true. For example we consider the function $f(x, y) = x$ with domain $\mathcal{D} = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$. Obviously the point $(1, 0)$ is the absolute maximum of the function f. This is the boundary point. On the other hand $\nabla f = (1, 0) \neq 0$. So we arrived to the contradiction.

Suppose we know all critial points of the function f. How to find among them the points of the local maximum and local minimum? The following theorem is very helpfull.

Theorem 2. Second Derivatives Test. *Suppose the second partial derivatives of the function f are continuous in a disk with center (a, b) and suppose that $\nabla f(a, b) = 0$. Let*

$$D = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2$$

Then

$$a) If \;\; D > 0 \;\; and \;\; f_{xx}(a, b) > 0 \;\; then \;\; (a, b) \;\; is \;\; a \;\; point \;\; of \;\; local\, minimum.$$

$$b) If \;\; D > 0 \;\; and \;\; f_{xx}(a, b) < 0 \;\; then \;\; (a, b) \;\; is \;\; a \;\; point \;\; of \;\; local\, maximum.$$

$$c) If \;\; D < 0 \;\; then \;\; (a, b) \;\; is \;\; not \;\; local \;\; maximum \;\; or \;\; minimum.$$

Example 1. For the function $f(x, y) = x^2 + y^2 + 4x - 6y$ find all critical points and classify them.

Solution. The partial derivatives of the function f are

$$f_x(x, y) = 2x + 4$$

$$f_y(x, y) = 2y - 6$$

Hence solving the system

$$\begin{cases} f_x(x, y) = 2x + 4 = 0 \\ f_y(x, y) = 2y - 6 = 0 \end{cases}$$

we obtain $x = -2, y = 3$. We got only one critical point $(-2, 3)$. Next we need the second order partial derivatives of the function f

$$f_{xx}(x, y) = 2, \quad f_{yy}(x, y) = 2, f_{xy}(x, y) = 0.$$

Hence $D = 4 > 0$ and $f_{xx}(-2, 3) > 0$. By the Second Derivative Test function $f(x, y)$ has at point $(-2, 3)$ local maximum. ■

Example 2. Find the absolute maximum or minimum for the function $f(x, y) = 2x^3 + y^4$ inside the disk $D = \{(x, y)|x^2 + y^2 \leq 1\}$.

Solution. First we look for the maximum or linimum inside the dist i.e. on the set $\{(x, y)|x^2+y^2 < 1\}$. In that case the point of maximum (minimum) should be the critical point of the function $f(x, y)$. The partial derivatives of the function f are

$$f_x(x, y) = 6x^2, \quad f_y(x, y) = 4y^3.$$

Hence solving the system

$$\begin{cases} f_x(x, y) = 6x^2 = 0 \\ f_y(x, y) = 4y^3 = 0 \end{cases}$$

we obtain $x = 0, y = 0$. Obviously $f_{xx}(0, 0) = f_{yy}(0, 0) = f_{xy}(0, 0) = 0$. So $D(0, 0) = 0$ and we can not apply the second derivative test. But still this point is the candidate for absolute maximum or minimum.

Next we look for a minimum or maximum of the function $f(x, y)$ on the circle $\{(x, y)|x^2 + y^2 = 1\}$. For the points on the circle the following is true

$$y^2 = 1 - x^2.$$

So if we plug-in into the formula for the function $f(x, y)$ instead of y^2 the function $1 - x^2$ we obtain

$$g(x) = 2x^3 + (1 - x^2)^2.$$

We need to find the maximum and minimum value of this function on the segment $[-1, 1]$. If $x = \pm 1$ then $y = 0$. We have $f(-1, 0) = -2, f(1, 0) = 2$. If function $g(x)$ has a minimum at some point inside of the segment $[-1, 1]$ by Fermat theorem

$$g'(x) = 6x^2 + 4x(x^2 - 1).$$

Solving the equation $g'(x) = 0$ we obtain the following solutions $x = 0$. If $x = 0$ then $y = \pm 1$. We have $f(0, 1) = f(0, -1) = 1$. Solving the equation

$$2(x^2 - 1) + 3x = 0.$$

We obtain two solutions $x = \frac{1}{2}$ and $x = -2$. Since $-2 \neq [-1, 1]$ we need to consider only $x = \frac{1}{2}$.

If $x = \frac{1}{2}$ for the second coordinate we have $y = \pm\frac{\sqrt{3}}{2}$. Then

$$f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = f(\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{25}{16}.$$

Hence the point $(-1, 0)$ is the absolute minimum and $(1, 0)$ is the absolute maximum.