

## INFINITE SEQUENCES AND SERIES

A sequence can be thought as a list of numbers written in a definite order

$$a_1, a_2, a_3, \dots a_n \dots$$

**Notation.**

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}.$$

**Definition..** A sequence has the limit  $L$  we write

$$\lim_{n \rightarrow +\infty} a_n = L.$$

IF for any  $\epsilon$  there exists  $N_0(\epsilon)$  such that

$$|L - a_n| \leq \epsilon$$

for all  $n \geq N(\epsilon)$ .

**Definition.** If a sequence  $\{a_n\}$  has a limit we say that this sequence is convergent.

**Theorem.** If  $\lim_{x \rightarrow +\infty} f(x) = L$  and  $f(n) = a_n$  when  $n$  is integer, then  $\lim_{n \rightarrow +\infty} a_n = L$ .

**Theorem.** If  $\{a_n\}$  and  $\{b_n\}$  are two convergent sequences and  $c$  is a constant then

$$\lim_{n \rightarrow +\infty} (a_n + b_n) = \lim_{n \rightarrow +\infty} a_n + \lim_{n \rightarrow +\infty} b_n.$$

$$\lim_{n \rightarrow +\infty} (a_n - b_n) = \lim_{n \rightarrow +\infty} a_n - \lim_{n \rightarrow +\infty} b_n.$$

$$\lim_{n \rightarrow +\infty} ca_n = c \lim_{n \rightarrow +\infty} a_n$$

$$\lim_{n \rightarrow +\infty} (a_n b_n) = \lim_{n \rightarrow +\infty} a_n \cdot \lim_{n \rightarrow +\infty} b_n$$

$$\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow +\infty} a_n}{\lim_{n \rightarrow +\infty} b_n} \quad \text{if} \quad \lim_{n \rightarrow +\infty} b_n \neq 0.$$

If the sequence is bounded above and below it is called bounded.

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-T}\mathcal{E}\mathcal{X}$

**Theorem.** If  $\lim_{n \rightarrow +\infty} |a_n| = 0$  then  $\lim_{n \rightarrow +\infty} a_n = 0$ .

**Definition.** A sequence  $\{a_n\}$  is called increasing if  $a_n < a_{n+1}$  for all  $n \geq 1$ . It is called decreasing if  $a_n > a_{n+1}$  for all  $n \geq 1$ . A sequence is monotonic if it is either increasing or decreasing.

**Definition.** A sequence  $\{a_n\}$  is bounded above if

$$a_n \leq M \quad \text{for all } n \geq 1.$$

A sequence  $\{a_n\}$  is bounded below if

$$a_n \geq M \quad \text{for all } n \geq 1.$$

**Theorem.** Every bounded monotonic sequence is convergent.

**Example 1.** Determine whether the sequence  $a_n = \frac{n^2-1}{n^2+1}$  converges or diverges and find the limit.

*Solution.*

$$\lim_{n \rightarrow +\infty} \frac{n^2-1}{n^2+1} = \lim_{n \rightarrow +\infty} \frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n^2}} = 1.$$

**Example 2.** Determine whether the sequence  $a_n = \sqrt{n+2} - \sqrt{n}$  converges or diverges and find the limit.

*Solution.*

$$\lim_{n \rightarrow +\infty} \sqrt{n+2} - \sqrt{n} = \lim_{n \rightarrow +\infty} \frac{2}{\sqrt{n+2} + \sqrt{n}} = 0.$$

**Example 3.** Determine whether the sequence  $a_n = \frac{1}{3n+5}$  is increasing, decreasing or not monotone.

*Solution.* We introduce the function  $f(x) = \frac{1}{3x+5}$ . Obviously  $f(n) = a_n$ . Next  $f'(x) = \frac{-3}{(3x+5)^2} < 0$  hence the function  $f(x)$  is decreasing for positive  $x$ . Therefore the sequence  $\{a_n\}$  is decreasing.

Suppose we have the sequence  $\{a_n\}$ . A sum

$$a_1 + a_2 + a_3 + a_4 + \dots$$

is called the infinite series.

**Notation**

$$\sum_{n=1}^{\infty} a_n.$$

**Definition.** Given series  $\sum_{n=1}^{\infty} a_n$ , let  $s_n$  denote the  $n$ th partial sum

$$s_n = \sum_{i=1}^n a_i$$

If the sequence  $s_n$  is convergent and  $\lim_{n \rightarrow +\infty} s_n = s$  the series  $\sum_{n=1}^{\infty} a_n$  is called convergent. Otherwise the series is called divergent.

**Definition.** The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1}$$

is convergent if  $|r| < 1$  and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If  $|r| > 1$  this series is divergent.

**Theorem.** If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Test for divergence.** If  $\lim_{n \rightarrow \infty} a_n$  does not exist or  $\lim_{n \rightarrow \infty} a_n \neq 0$  then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.

**Theorem.** If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are convergent series, then so the series  $\sum_{n=1}^{\infty} (a_n + b_n)$  and  $\sum_{n=1}^{\infty} (a_n - b_n)$  and  $\sum_{n=1}^{\infty} ca_n$  Moreover

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$$

**Example 3.** Determine either or not the infinite series  $\sum_{n=1}^{\infty} \frac{n}{\sqrt{1+n^2}}$  is convergent.

*Solution.* Since the limit

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{1+n^2}} = 1.$$

the infinite series is divergent.