LECTURE 7 (267)

Second order ordinary differential equations.

By the second order ordinary differential equation we mean an equation of the form

$$G(t, y, y', y'') = 0.$$

In this section we concentrate on the **linear** ordinary differential equations. By the linear second order differential equation we mean the equation of the form

$$A(t)y'' + B(t)y' + C(t)y = F(t). \tag{1}$$

If $F(t) \equiv 0$ the equation is called **homogeneous** otherwise it is called **inhomogeneous.**

If $A(t) \neq 0$ then we can rewrite (1) in the form

$$y'' + p(t)y' + q(t)y = f(t). \tag{2}$$

Theorem 1. *If $y_1(t)$ and $y_2(t)$ are solutions to equation $y'' + p(t)y' + q(t)y = f(t)$ the linear combination of these functions*

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is the solution to this equation also.

Next we consider the initial value problem for equation(2)

Theorem 2. *Suppose that the functions p, q and f are continuous on the open interval I containing the point a. Then given two any numbers b_0 and b_1 the equation*

$$y'' + p(t)y' + q(t)y = f(t). \tag{3}$$

has a unique solution on the interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1.$$

Definition. *Two functions f_1 and f_2 are linearly dependant on the interval I if there exist constants c_1 and c_2 such that*

$$c_1 f_1(t) + c_2 f_2(t) = 0$$

for all $t \in I$.

The homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{4}$$

always has two linearly independent solutions $y_1(t)$ and $y_2(t)$. The general solution to equation (4) given by formula

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

where the functions y_1, y_2 is the pair of linearly independent solutions to equation (4).

Definition. *Let f and g be the differentiable functions on the interval I. Then Wronskian of these functions given by formula*

$$W(f, g) = fg' - f'g.$$

Theorem. *Let functions y_1 and y_2 be solutions to homogeneous equation (4). Then A) If y_1 and y_2 are linearly dependant on the interval I then $W(y_1, y_2) = 0$ on I. B) If y_1 and y_2 are linearly independent on the interval I then $W(y_1, y_2) \neq 0$ at each point of I.*

Linear Second-Order Equation with Constant Coefficients.

We consider the equation

$$ay'' + by' + cy = 0. \tag{5}$$

The polynomial $ar^2 + br + c$ is called the characteristic polynomial of this equation. Let r_1, r_2 be the roots of characteristic polynomial i.e. solutions to the equation

$$ar^2 + br + c = 0.$$

Case A. Roots r_1 and r_2 are real and $r_1 \neq r_2$ then equation (5) has two linearly independent solutions

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}.$$

The general solution is

$$Y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Case B. Roots r_1 and r_2 are real and $r_1 = r_2 = r$. Then equation (5) has two linearly independent solutions

$$y_1(t) = e^{rt}, \quad y_2(t) = te^{rt}.$$

The general solution is

$$Y(t) = c_1 e^{rt} + c_2 te^{rt}.$$

Case 3. Roots r_1 and r_2 are complex. Then $r_1 = \tilde{a} + i\tilde{b}$ and $r_2 = \tilde{a} - i\tilde{b}$. Equation(5) has two linearly independent solutions

$$y_1(t) = e^{\tilde{a}t} cos(\tilde{b}t), \quad y_2(t) = e^{\tilde{a}t} sin(\tilde{b}t).$$

The general solution is

$$Y(t) = c_1 e^{\tilde{a}t} cos(\tilde{b}t) + c_2 e^{at} sin(\tilde{b}t).$$

Example 1. Find a general solution to the ordinary differential equation

$$2y'' + 3y' = 0.$$

Solution. The characteristic polynomial is $2r^2 + 3r = 0$. The roots are $r_1 = 0, r_2 = -\frac{3}{2}$. Then the general solution is

$$Y(t) = c_1 + c_2 e^{-\frac{3}{2}t}.$$

Example 2. Find a general solution to the ordinary differential equation

$$y'' - 3y' + 2y = 0.$$

Solution. The characteristic polynomial is $r^2 - 3r + 2 = 0$. The roots are $r_1 = 1, r_2 = 2$. Then the general solution is

$$Y(t) = c_1 e^{t} + c_2 e^{2t}.$$

Example 3. Find a general solution to the ordinary differential equation

$$y'' - 4y' + 4y = 0.$$

Solution. The characteristic polynomial is $r^2 - 4r + 4 = 0$. The roots are $r_1 = 2, r_2 = 2$. Then the general solution is

$$Y(t) = c_1 te^{2t} + c_2 e^{2t}.$$