

LECTURE 8 (FALL 2003)

Variable Gravitational Acceleration.

According to the Newton's law of gravitation the gravitational force of attraction between two point masses M and m is

$$F = G \frac{Mm}{r^2},$$

where G is a certain empirical constant $G = 6.6726 \times 10^{-11} N \cdot (m/kg)^2$ in mks units.

Example 1 *Escape velocity.*

What initial velocity v_0 is necessary for projectile to escape from the earth altogether.

Solution. Denote by R the radius of the earth, let v_0 be the initial velocity. We put the center of our coordinate system into the center of the earth. Then according to the Newton's law

$$\frac{dv}{dt} = -\frac{GM}{r^2}.$$

Then

$$\frac{1}{2}v^2 = \frac{GM}{r} + C$$

Hence

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right).$$

We should find a sufficient condition such that $v^2(t) > 0$ for all t . Obviously

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right) > v_0^2 - \frac{2GM}{R}.$$

If

$$v_0^2 - \frac{2GM}{R} \geq 0$$

So

$$v_0 = \sqrt{\frac{2GM}{R}}.$$

Example 2 *Escape velocity.*

What initial velocity v_0 is necessary for projectile to escape from the earth altogether if the object is attracted both by earth and the moon. the distance between centers of the earth and the moon is $S = 384400km$.

Solution. According to the Newton's law

$$\frac{d^2r}{dt^2} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}, r(0) = R, r'(0) = v_0$$

Then

$$\frac{v^2}{2} = \frac{GM_e}{r} + \frac{GM_m}{(S-r)} + C.$$

Obviously

$$C = \frac{v_0^2}{2} - \left(\frac{GM_e}{R} + \frac{GM_m}{(S-R)} \right).$$

$$\frac{v^2}{2} = \frac{GM_e}{r} + \frac{GM_m}{(S-r)} + \frac{v_0^2}{2} - \left(\frac{GM_e}{R} + \frac{GM_m}{(S-R)} \right).$$

Denote

$$f(r) = \frac{GM_e}{r} + \frac{GM_m}{(S-r)} + \frac{v_0^2}{2} - \left(\frac{GM_e}{R} + \frac{GM_m}{(S-R)} \right)$$

we are looking for the minimum of this function on the segment $[R, S]$. Solving the equation $f'(r) = 0$ we have

$$r_0 = \frac{S\sqrt{M_e}}{\sqrt{M_e} + \sqrt{M_m}}.$$

we should have

$$f(r_0) > 0$$

So

$$\frac{G}{S}(\sqrt{M_e} + \sqrt{M_m})^2 + \frac{v_0^2}{2} - \left(\frac{GM_e}{R} + \frac{GM_m}{(S-R)} \right) > 0$$

Hence

$$v_0 > 2\sqrt{\frac{G}{S}(\sqrt{M_e} + \sqrt{M_m})^2 - \left(\frac{GM_e}{R} + \frac{GM_m}{(S-R)} \right)}.$$

Now we consider several models where we take into account the air resistance. First we assume that the force of air resistance is proportional to the square of velocity:

$$F_R = \pm kv^2.$$

The Newton's law gives

$$m \frac{dv}{dt} = F_G + F_R = -mg - kv|v|$$

or

$$\frac{dv}{dt} = -g - \rho v|v|.$$

Upward motion. We set

$$C_1 = \arctan\left(v_0 \sqrt{\frac{\rho}{g}}\right)$$

Then

$$v(t) = \sqrt{\frac{g}{\rho}} \tan(C_1 - t\sqrt{\rho g})$$

$$y(t) = y_0 + \frac{1}{\rho} \ln \left| \frac{\cos(C_1 - t\sqrt{\rho g})}{\cos C_1} \right|$$

Downward motion. We set

$$C_2 = \tanh^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right)$$

$$v(t) = \sqrt{\frac{g}{\rho}} \tanh(C_2 - t\sqrt{\rho g})$$

$$y(t) = y_0 + \frac{1}{\rho} \ln \left| \frac{\cosh(C_2 - t\sqrt{\rho g})}{\cosh C_2} \right|$$

Example 3. Consider a body that moves horizontally through a medium whose resistance is proportional to the square of velocity v so that $\frac{dv}{dt} = -kv^2$. Show what

$$v(t) = \frac{v_0}{1 + v_0 kt}$$

and that

$$x(t) = x_0 + \frac{1}{k} \ln(1 + v_0 kt)$$

Solution. The equation

$$\frac{dv}{dt} = -kv^2 = -\frac{k}{v^2}$$

is separable. We have

$$g(t) = -k, \quad \int g(t)dt = -kt, \quad f(v) = \frac{1}{v^2}, \quad \int f(v)dv = -\frac{1}{v}.$$

Then

$$-\frac{1}{v(t)} = -kt + C.$$

From this equation assuming that $v(0) = v_0$ we have

$$-\frac{1}{v(t)} = -kt - \frac{1}{v_0}.$$

So

$$v(t) = \frac{v_0}{1 + v_0kt}.$$

Next we recall that $\frac{dx}{dt} = v(t)$. Hence

$$\frac{dx}{dt} = \frac{v_0}{1 + v_0kt}$$

Then

$$x(t) = C + \frac{1}{k} \ln(1 + v_0kt).$$

Assuming that $x(0) = x_0$ we have

$$x(t) = x_0 + \frac{1}{k} \ln(1 + v_0kt).$$

Example 4 A motorboat weights 3200lb and its motor provides a thrust of 500 lb. Assume that the water resistance is 100 pounds for each foot per second of the speed v of the boat. Then

$$1000 \frac{dv}{dt} = 5000 - 100v$$

If the boat starts from rest what is the maximum velocity that it can attain?