LECTURE 7 (267)

Population models

Suppose that we have some population of animals. Assume that the population changes only by occurrence of births and deaths.

We introduce the *birth* and *death* rates

$\beta(t)$ is the number of births per unit of population per unit of time.

$\delta(t)$ is the number of deaths per unit of population per unit of time.

Let $P(t)$ be a number of species in the population in time moment t. Then the dynamics of the population is described by the following equation:

$$\frac{dP}{dt} = (\beta - \delta)P. \tag{1}$$

Logistic equation.

Assume that the birth rate is decreasing function of the population :

$$\beta(t) = \beta_0 - \beta_1 P.$$

where β_0, β_1 are some positive constants and the death rate is constant : $\delta(t) = \delta_0$.

Then equation (1) has form

$$\frac{dP}{dt} = kP(M - P), \tag{2}$$

where $k = \beta_0 - \delta_0, M = \frac{\beta_0 - \delta_0}{\beta_1}$.

Solving this equation we arrive to the following formula

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}.$$

Obviously

$$\lim_{t \to +\infty} P(t) = M.$$

The number M is called the *limiting population.*

Next we consider the population of "unsophisticated " animals in which females rely solely on chance encounters to meet males for reproductive purposes. The birth rate for such population is $\beta = kP$. Assume that the death δ rate is constant. Then

$$\frac{dP}{dt} = kP(P - M).$$

Here $M = \frac{\delta}{k}$. We try to solve this equation. First we assume

$$P_0 > M.$$

Then

$$P(t) = \frac{CMe^{-kMt}}{Ce^{-kMt} - 1}.$$

where $C = \frac{P_0}{P_0 - M}$.

Next we assume that

$$0 < P_0 < M.$$

In this case

$$P(t) = \frac{CMe^{-kMt}}{Ce^{-kMt} + 1},$$

where $C = \frac{P_0}{M - P_0}$.

Example 1. Solve the initial value problem

$$\frac{dx}{dt} = 3x(5 - x), \quad x(0) = 8.$$

Solution. This equation is separable equation:

$$\frac{dx}{dt} = 3x(5 - x) = \frac{3}{(\frac{1}{x(5-x)})}.$$

Then

$$g(t) = 3, \quad \int g(t)dt = 3t, \quad f(x) = \frac{1}{x(5 - x)},$$

$$\int f(x)dx = \frac{1}{5}\int (\frac{1}{x} - \frac{1}{x - 5})dx = \frac{1}{5}(\ln x - \ln(x - 5)) = \frac{1}{5}\ln\frac{x}{x - 5}.$$

Therefore we have

$$\frac{1}{5}\ln\frac{x}{x - 5} = 3t + C. \tag{3}$$

Using the initial condition we find the constant C:

$$\frac{1}{5}\ln\frac{x(0)}{x(0)-5}=\frac{1}{5}\ln\frac{8}{3}=C.$$

Now we can rewrite formula (3) in the form

$$\frac{1}{5}\ln\frac{x}{x-5}=3t+\ln\frac{8}{3}.$$

From this equation we obtain

$$\frac{x(t)}{x(t)-5}=\frac{8}{3}e^{15t}.$$

Solving this equation respect to $x(t)$ we arrive to the formula

$$x(t)=\frac{\frac{40}{3}e^{15t}}{\frac{8}{3}e^{15t}-1}.$$

Example 2. Suppose that when the lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$. (a) Show that

$$P(t)=(\frac{1}{2}kt+\sqrt{P_0})^2.$$

where k is a constant.

(b) If $P_0=100$ then and after six months there are 169 fish in the lake, how many will be after one year?

Solution. We measure the time in months. Since β and δ are both inversely proportional to $\sqrt{P}$ there exist such a constants k_1, k_2 such that

$$\beta=\frac{k_1}{\sqrt{P}},\quad \delta=\frac{k_2}{\sqrt{P}}.$$

From equation (1) we obtain

$$\frac{dP}{dt}=k\sqrt{P},$$

where $k=k_1-k_2$. This equation is separable and the general solution is

$$2\sqrt{P(t)}=kt+C.$$

If $P(0) = P_0$ from this equation we obtain

$$2\sqrt{P(t)} = kt + 2\sqrt{P_0}.$$

Hence

$$P(t) = (\frac{kt}{2} + \sqrt{P_0})^{\frac{1}{2}}. \quad (3)$$

Now if $P_0 = 100$ formula (3) has the form

$$P(t) = (\frac{kt}{2} + 10)^{\frac{1}{2}}.$$

Since $P(6) = 169$ we have

$$169 = (3k + 10)^{\frac{1}{2}}.$$

Hence $k = 1$ then $P(12) = 16^2$.