LECTURE 6 (267)

Exact differential equations

We consider the ordinary differential equation

$$M(x,y)+N(x,y)\frac{dy}{dx}=0. \tag{1}$$

We say that the O.D.E (1) is exact if

$$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}. \tag{2}$$

If condition (2) holds true there exists a function $F(x,y)$ such that

$$N(x,y)=F_y(x,y), \quad M(x,y)=F_x(x,y). \tag{3}$$

If O.D.E. is exact and $y(x)$ is some solution to equation (1) there exist a constant C (which in general depends on y) such that

$$F(x,y)=C \tag{4}$$

Example 1. Solve the ordinary differential equation

$$(1+ye^{xy})+(2y+xe^{xy})\frac{dy}{dx}=0.$$

Solution. We have $M(x,y)=1+ye^{xy}$ and $N(x,y)=2y+xe^{xy}$. Then $M_y=e^{xy}+xye^{xy}$ and $N_x=e^{xy}+xye^{xy}$. So $N_x=M_y$ and our equation is exact. Next we try to find a function $F(x,y)$ such that equality (3) holds true. We have

$$F_x=1+ye^{xy} \quad \textit{and} \quad F_y=2y+xe^{xy}.$$

Therefore

$$F(x,y)=x+e^{xy}+C_1(y) \quad \textit{and} \quad F(x,y)=y^2+e^{xy}+C_2(x).$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

We get rid of $F(x,y)$

$$x + e^{xy} + C_1(y) = y^2 + e^{xy} + C_2(x).$$

Or

$$x - C_2(x) = y^2 - C_1(y). \tag{5}$$

The right hand side of (5) is the x dependent function and the right hand side is y dependent function. Since x and y are independent variables the equality is possible if and only if there exist a constant C such that

$$x - C_2(x) = y^2 - C_1(y) = C$$

From this equation we find $C_1(y) = y^2$. So by formula (4)

$$x + e^{xy(x)} + y^2(x) = C.$$

Example 2. Solve the ordinary differential equation

$$(cos(x) + \ln y) + (\frac{x}{y} + e^y)\frac{dy}{dx} = 0.$$

Solution. We have $M(x,y) = cos(x) + \ln y$ and $N(x,y) = \frac{x}{y} + e^y$. Then $M_y = \frac{1}{y}$ and $N_x = \frac{1}{y}$. So the equation is exact. Next we try to find a function $F(x,y)$ such that equality (3) holds true. We have

$$F_x = cos(x) + \ln y \quad and \quad F_y = \frac{x}{y} + e^y.$$

Then

$$F = sin(x) + x\ln y + C_1(y) \quad and \quad F = x\ln y + e^y + C_2(x).$$

We get rid of $F(x,y)$

$$sin(x) + x\ln y + C_1(y) = x\ln y(t) + e^y + C_2(x)$$

So

$$sin(x) - C_2(x) = e^y - C_1(y). \tag{6}$$

The right hand side of (6) is the x dependent function and the right hand side is y dependent function. Since x and y are independent variables the equality is possible if and only if there exist a constant C such that

$$sin(x) - C_2(x) = e^y - C_1(y) = C.$$

Then

$$F(x,y) = sin(x) + x \ln y + e^y.$$

Therefore

$$sin(x) + x + \ln y(x) + e^{y(x)} = C$$

Homogeneous equations.

The first order ordinary differential equation

$$\frac{dy}{dt} = F(\frac{y}{t}) \tag{7}$$

is called homogeneous.

We introduce the function $v(t)$ by the formula

$$v(t) = \frac{y(t)}{t}.$$

This function satisfies the O.D.E.

$$t\frac{dy}{dt} = F(v) - v \tag{8}$$

Equation (8) is separable.

Example 3. Find a general solution to the ordinary differential equation

$$\frac{dy}{dt} = \frac{t+3y}{y-3t}.$$

Solution. This equation is homogeneous. Really

$$\frac{dy}{dt} = \frac{t+3y}{y-3t} = \frac{1+3\frac{y}{t}}{\frac{y}{t}-3}.$$

So

$$F(z) = \frac{1+3z}{z-3}.$$

Note that $F(z) - z = \frac{1-z^2+6z}{z-3}$. We introduce the function $v(t) = \frac{y(t)}{t}$.

Then by (8) we have

$$t\frac{dv}{dt} = \frac{1-v^2+6v}{v-3} \tag{9}$$

This equation is separable

$$f(v) = \frac{v-3}{1-v^2+6v}, \quad g(t) = \frac{1}{t}.$$

So

$$\int f(v)dv = -\frac{1}{2}\ln|1-v^2+6v| \quad and \int \frac{1}{t}dt = \ln|t|.$$

Hence

$$-\frac{1}{2}\ln|1-v^2+6v| = \ln|t| + C. \tag{9}$$

Now we return in (9) to function $y(t)$:

$$-\frac{1}{2}\ln|1-(\frac{y(t)}{t})^2 + 6\frac{y(t)}{t}| = \ln|t| + C$$

Find a general solution to the ordinary differential equation

$$\frac{dy}{dt} = -\frac{3t^2+2y^2}{4ty}.$$

Solution. This equation is homogeneous. Really

$$\frac{dy}{dt} = -\frac{3t^2+2y^2}{4ty} = -\frac{3+2(\frac{y}{t})^2}{4\frac{y}{t}}. \tag{10}$$

Hence $F(z) = -\frac{3+2z^2}{4z}$ and $F(z) - z = -\frac{3+6z^2}{4z}$. We introduce the function $v(t) = \frac{y(t)}{t}$. Then by (8) we have

$$t\frac{dv}{dt} = -\frac{3+6v^2}{4v} \tag{11}$$

This equation is separable

$$f(v) = -\frac{4v}{3+6v^2}, \quad g(t) = \frac{1}{t}.$$

So

$$\int f(v)dv = -\frac{1}{3}\ln(3+6v^2) \quad and \int \frac{1}{t}dt = \ln|t|.$$

Hence

$$-\frac{1}{3}\ln(3+6v^2) = \ln|t| + C.$$

Now we return to the function $y(t)$

$$-\frac{1}{3}\ln(3+6(\frac{y(t)}{t})^2) = \ln|t| + C.$$