LECTURE 4(267)

According to Newton's law of cooling the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$. That is

$$\frac{dT}{dt} = k(A - T),$$

where k is positive constant.

Example 1. A pitcher of butter milk initially at 25^0C is to be cooled by setting it on the front porch, where the temperature is 0^0C. Suppose that the temperature of the buttermilk has dropped to 15^0C after 20 min. When it will be at 5^0C?

Solution. We note that $A = 0$, $T(0) = 25$, $T(20) = 15$. Denote by t_0 the time moment when $T(t_0) = 5$. Then by Newton's law

$$\frac{dT}{dt} = -kT.$$

The general solution to this equation is $T(t) = Ce^{-kt}$. Thanks to the initial condition $T(0) = 25$ we have $C = 25$ and $T(t) = 25e^{-kt}$. Next we find the constant k. Since $T(20) = 15 = 25e^{-20k}$ we have $k = \frac{\ln \frac{5}{3}}{20}$. The formula for the temperature is

$$T(t) = 25e^{-\frac{\ln \frac{5}{3}}{20}t}.$$

Now we are able to find t_0:

$$T(t_0) = 5 = 25e^{-\frac{\ln \frac{5}{3}}{20}t_0}.$$

From this equation we obtain immediately

$$t_0 = \frac{20 \ln 5}{\ln \frac{5}{3}}.$$

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Population Growth. Suppose that $P(t)$ is the number of individuals in a population having constant birth and death rates β and δ. Then the dynamics of a population can be described by the ordinary differential equation

$$\frac{dP}{dt} = kP(t)$$

where $k = \beta - \delta$.

Example 2. In a certain culture of bacteria the number of bacteria increased sixfold in 10 h. How long did it take for the population to double?

Solution. Let $P(t)$ be a number of bacteria at moment t. Then

$$\frac{dP}{dt} = kP$$

Denote by P_0 the number of bacteria at moment $t = 0$. Then $P(t) = P_0 e^{kt}$. Since $P(10) = 6P_0$ we have

$$6P_0 = P_0 e^{10k}$$

This formula implies immediately that

$$k = \frac{\ln 6}{10}.$$

Hence

$$P(t) = P_0 e^{\frac{\ln 6}{10} t}$$

Let t_0 be a moment of time such that $P(t_0) = 2P_0$. Then

$$2P_0 = P_0 e^{\frac{\ln 6}{10} t_0}$$

and $t_0 = \frac{10 \ln 2}{\ln 6}$.

Radioactive decay. Consider the sample of a certain material that contains $N(t)$ atoms of a certain radioactive isotope at time t. It has been observed that a constant fraction of those radioactive atoms will spontaneously decay during each unit of time. We describe this process by the ordinary differential equation

$$\frac{dN}{dt} = -kN.$$

Here positive constant k depends on particular radioactive isotope.

Example 3. The half-life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident has left the level of cobalt radiation in a certain region at 100 times the level acceptable for human habitation. How long will it be until the region is again habitable?

Solution. Denote by $N(t)$ the mass of radioactive cobalt at moment t. Then

$$\frac{dN}{dt} = -kN$$

If N_0 is the mass of radioactive cobalt at moment $t = 0$ then

$$N(t) = N_0 e^{-kt}$$

Since $N(5.27) = \frac{N_0}{2}$ we have

$$N(5.27) = \frac{N_0}{2} = N_0 e^{-5.27k}.$$

Then $k = \frac{\ln 2}{5.27}$. Hence

$$N(t) = N_0 e^{-\frac{\ln 2}{5.27}t}$$

The region will be habitable when only 99 procent of cobalt decay. Denote by t_0 the time moment when this happen. Then

$$N(t_0) = \frac{N_0}{100} = N_0 e^{-\frac{\ln 2}{5.27}t_0}$$

This formula implies

$$t_0 = \frac{5.27 \ln 100}{\ln 2}.$$

Torricelli's law Suppose a tank with water has a hole at its bottom. Denote by $y(t)$ the level of a water in the tank at moment t. Then the function $y(t)$ satisfies the ordinary differential equation

$$A(y(t))\frac{dy}{dt} = -k\sqrt{y},$$

Here $A(y)$ is the horizontal cross sectional area of the tank and k is some positive constant.

Example 4. At time $t = 0$ the bottom plug of a full conical water tank 16 ft high is removed. After one hour the water in the tank is 9 ft deep. When the tank will be empty?

Solution. Denote by $y(t)$ the height of a water in a tank. Then

$$A(y(t))\frac{dy}{dt} = -k\sqrt{y}$$

Note that

$$A(y(t)) = \pi r(t)^2 = \pi L y^2(t)$$

Then

$$y^2\frac{dy}{dt} = \mu\sqrt{y},$$

where $\mu = -k/(\pi L)$.

This equation is separable,so

$$y^{\frac{5}{2}}(t) = \mu t + C$$

Using the initial condition $y(0) = 16$ we obtain

$$y^{\frac{5}{2}}(0) = 4^5 = C$$

So

$$y^{\frac{5}{2}}(t) = \mu t + 4^5.$$

Since $y(1) = 9$ we have

$$y^{\frac{5}{2}}(1) = 3^5 = \mu + 4^5$$

We have

$$\mu = 3^5 - 4^5$$

Denote by T_0 the time moment when $y(t_0) = 0$ Then

$$y^{\frac{5}{2}}(t_0) = 0 = (3^5 - 4^5)t_0 + 4^5.$$

Hence

$$t_0 = \frac{4^5}{4^5 - 3^5}.$$