LECTURE 34 (267)

Homogeneous systems of linear ordinary differential equations.

Example 1. Find a general solution to the system

$$x_1' = 4x_1 + x_2 + 4x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 4x_1 + x_2 + 4x_3.$$

Solution. First we rewrite the system in the form

$$x' = Ax,$$

where

$$A = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A. We set

$$A - \lambda E = \begin{pmatrix} 4-\lambda & 1 & 4 \\ 1 & 7-\lambda & 1 \\ 4 & 1 & 4-\lambda \end{pmatrix}.$$

Therefore

$$det(A - \lambda E) = -\lambda(\lambda^2 - 15\lambda + 54).$$

Hence we have three eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = 9, \quad \lambda_3 = 6.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 0$. We have

$$A\vec{e} = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix} \vec{e} = 0.$$

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This equation is equavalent to the following system

$$\begin{cases} 4e_1 + e_2 + 4e_3 = 0 \\ e_1 + 7e_2 + e_3 = 0 \\ 4e_1 + e_2 + 4e_3 = 0 \end{cases}$$

We can easily observe that the first and the third equations in this system are exactly the same. So

$$\begin{cases} \\ e_1 + 7e_2 + e_3 = 0 \\ 4e_1 + e_2 + 4e_3 = 0 \end{cases}$$

Subtracting from the second equation the first one multiplied by -4 we obtain

$$-27e_2 = 0.$$

So $e_2 = 0$ and then $e_1 + e_3 = 0$. Therefore we may take $e_3 = -1$ and $e_1 = 0$.

The the eigenvector associated with the eigenvalue $\lambda = 0$ equal $\vec{e} = (1, 0, -1)$.

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 9$. We have

$$(A - 9E)\vec{e} = \begin{pmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{pmatrix} \vec{e} = 0.$$

This equation is equivalent to the following system

$$\begin{cases} -5e_1 + e_2 + 4e_3 = 0 \\ e_1 - 2e_2 + e_3 = 0 \\ 4e_1 + e_2 - 5e_3 = 0 \end{cases}$$

We can rewrite this system in the form

$$\begin{cases} e_1 - 2e_2 + e_3 = 0 \\ 4e_1 + e_2 - 5e_3 = 0 \\ -5e_1 + e_2 + 4e_3 = 0 \end{cases}$$

We multiply the first equation by -4 and add it to the second:

$$\begin{cases} e_1 - 2e_2 + e_3 = 0 \\ 9e_2 - 9e_3 = 0 \\ -5e_1 + e_2 + 4e_3 = 0 \end{cases}$$

Next we multiply the first equation by 5 and add it to the third

$$\begin{cases} e_1 - 2e_2 + e_3 = 0 \\ 9e_2 - 9e_3 = 0 \\ -9e_2 + 9e_3 = 0 \end{cases}$$

Therefore $e_2 = e_3$ and then the first equation has the form

$$e_1 - e_3 = 0.$$

Hence $\vec{e} = (1, 1, 1)$.

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 6$. We have

$$(A - 6E)\vec{e} = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{pmatrix} \vec{e} = 0.$$

This equation is equivalent to the following system

$$\begin{cases} -2e_1 + e_2 + 4e_3 = 0 \\ e_1 + e_2 + e_3 = 0 \\ 4e_1 + e_2 - 2e_3 = 0 \end{cases}$$

We can rewrite this system in the form

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ -2e_1 + e_2 + 4e_3 = 0 \\ 4e_1 + e_2 - 2e_3 = 0 \end{cases}$$

We multiply the first equation by 2 and add it to the second:

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ 3e_2 + 6e_3 = 0 \\ 4e_1 + e_2 - 2e_3 = 0 \end{cases}$$

Next we multiply the first equation by -4 and add it to the third

$$\begin{cases} e_1 + e_2 + e_3 = 0 \\ 3e_2 + 6e_3 = 0 \\ -3e_2 - 6e_3 = 0 \end{cases}$$

Therefore $e_2 = -2e_3$ and then the first equation has the form

$$e_1 - e_3 = 0.$$

Hence $\vec{e} = (1, -2, 1)$. Then the general solution is

$$X(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{9t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{6t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Now we consider the case when a matrix A has one real eigenvalue and two complex.

Example 2. Find a general solution to the system

$$x_1' = 5x_1 + 5x_2 + 2x_3, \quad x_2' = -6x_1 - 6x_2 - 5x_3, \quad x_3' = 6x_1 + 6x_2 + 5x_3.$$

Solution. First we rewrite the system in the form

$$x' = Ax,$$

where

$$A = \begin{pmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A. We set

$$A - \lambda E = \begin{pmatrix} 5-\lambda & 5 & 2 \\ -6 & -6-\lambda & -5 \\ 6 & 6 & 5-\lambda \end{pmatrix}.$$

Therefore

$$det(A - \lambda E) = -\lambda(\lambda^2 - 4\lambda + 13).$$

Hence we have three eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = 2 + 3i, \quad \lambda_3 = 2 - 3i.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 0$. We have

$$A\vec{e} = \begin{pmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{pmatrix} \vec{e} = 0.$$

This equation is equivalent to the following system

$$\begin{cases} 5e_1 + 5e_2 + 2e_3 = 0 \\ -6e_1 - 6e_2 - 5e_3 = 0 \\ 6e_1 + 6e_2 + 5e_3 = 0 \end{cases}$$

We can easily observe that the second and the third equations in this system are exactly the same. So

$$\begin{cases} 5e_1 + 5e_2 + 2e_3 = 0 \\ -6e_1 - 6e_2 - 5e_3 = 0 \end{cases}$$

adding to the second equation the first one multiplied by $\frac{6}{5}$ we obtain

$$(-5 + \frac{12}{5})e_3 = 0$$

So $e_3 = 0$ and then $e_1 + e_2 = 0$. Therefore we may take $e_2 = -1$ and $e_1 = 1$.

The the eigenvector associated with the eigenvalue $\lambda = 0$ equal $\vec{e} = (1, -1, 0)$.

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 2 + 3i$. We have

$$(A - (2+3i)E)\vec{e} = \begin{pmatrix} 3-3i & 5 & 2 \\ -6 & -8-3i & -5 \\ 6 & 6 & 3-3i \end{pmatrix} \vec{e} = 0.$$

This equation is equivalent to the following system

$$\begin{cases} (3-3i)e_1 + 5e_2 + 2e_3 = 0 \\ -6e_1 - (8+3i)e_2 - 5e_3 = 0 \\ 6e_1 + 6e_2 + (3-3i)e_3 = 0 \end{cases}$$

Multiplying the first equation by $1 + i$ we obtain

$$\begin{cases} 6e_1 + (5+5i)e_2 + (2+2i)e_3 = 0 \\ -6e_1 - (8+3i)e_2 - 5e_3 = 0 \\ 6e_1 + 6e_2 + (3-3i)e_3 = 0 \end{cases}$$

Next we add to the second equation the third one and subtract from the first equation the third equation:

$$\begin{cases} (-1+5i)e_2 + (-1+5i)e_3 = 0 \\ -(2+3i)e_2 - (2+3i)e_3 = 0 \\ 2e_1 + 2e_2 + (1-1i)e_3 = 0 \end{cases}$$

This system is equivalent to the system

$$\begin{cases} e_2 + e_3 = 0 \\ e_2 + e_3 = 0 \\ 2e_1 + 2e_2 + (1 - 1i)e_3 = 0 \end{cases}$$

Then we have

$$e_2 = -e_3, \quad 2e_1 + 2e_2 + (1 - i)e_3 = 0.$$

Therefore $2e_1 = (1 + i)e_3$. We set $e_3 = 2$. Then $e_2 = -2, e_1 = 1 + i$. Finally we obtain

$$\vec{e} = (1, -2, 2) + (1, 0, 0)i.$$

The general solution is

$$X(t) = C_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{2t} \left(\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} cos(3t) - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} sin(3t) \right) + C_3 e^{2t} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} cos(3t) + \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} sin(3$$