LECTURE 34 (267)

Homogeneous systems of linear ordinary differential equations.

Example 1. Find a general solution to the system

$$x_1' = 5x_1 - 9x_2, \quad x_2' = 2x_1 - x_2.$$

Solution. First we rewrite the system in the form

$$x' = Ax$$

where

$$A = \begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A. We set

$$A - \lambda E = \begin{pmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{pmatrix}.$$

Therefore

$$det(A - \lambda E) = \lambda^2 - 4\lambda - 13.$$

Hence we have two eigenvalues

$$\lambda_{1,2} = 2 \pm 3i.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 2 + 3i$. We have

$$A - (2+3i)E = \begin{pmatrix} 2-3i & -9 \\ 2 & -3-3i \end{pmatrix}.$$

Hence the system $(A - (2+3i)E)\vec{e} = 0$ is equivalent to equation $2e_1 - (3 + 3i)e_2 = 0$. The solution to this equation is $\vec{e} = (3+3i, 2)$. Now we are using the following theorem from the previous lecture

Theorem 2. *If eigenvalue $\lambda = p+iq$ is the complex number and $V = \vec{a}+i\vec{b}$ is an eigenvector associated with this eigenvalue then the functions $x_1(t), x_2(t)$:*

$$x_1(t) = e^{pt}(\vec{a}cos(qt) - \vec{b}sin(qt)), \quad x_2(t) = e^{pt}(\vec{b}cos(qt) + \vec{a}sin(qt))$$

are solutions to the system homogeneous system $\frac{dx}{dt} = Ax$.

Using this Theorem 2 with $p = 2$ and $q = 3$ we have

$$x_1(t) = e^{2t}\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix} cos(3t) - \begin{pmatrix} 3 \\ 0 \end{pmatrix} sin(3t)\right),$$

$$x_2(t) = e^{2t}\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix} cos(3t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} sin(3t)\right).$$

Then the general solution is

$$X(t) = C_1x_1(t) + C_2x_2(t).$$

Example 2. Find a general solution to the system

$$x_1' = x_1 - 5x_2, \quad x_2' = x_1 + 3x_2.$$

Solution. First we rewrite the system in the form

$$x' = Ax$$

where

$$A = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A. We set

$$A - \lambda E = \begin{pmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{pmatrix}.$$

Therefore

$$det(A - \lambda E) = \lambda^2 - 4\lambda + 8.$$

Hence we have two eigenvalues

$$\lambda_{1,2} = 2 \pm 2i.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 2 + 2i$. We have

$$A - (2+2i)E = \begin{pmatrix} -1-2i & -5 \\ 1 & 1-2i \end{pmatrix}.$$

Hence the system $(A - (2+2i)E)\vec{e} = 0$ is equivalent to equation $e_1 + (1 - 2i)e_2 = 0$. The solution to this equation is $\vec{e} = (2i-1, 1)$. Using this Theorem 2 with $p = 2$ and $q = 2$ we have

$$x_1(t) = e^{2t}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} cos(2t) - \begin{pmatrix} 2 \\ 0 \end{pmatrix} sin(2t)\right),$$

$$x_2(t) = e^{2t}\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} cos(2t) + \begin{pmatrix} -1 \\ 1 \end{pmatrix} sin(2t)\right).$$

Then the general solution is

$$X(t) = C_1x_1(t) + C_2x_2(t).$$

Example 3. Find a general solution to the system

$$x_1' = 7x_1 - 5x_2, \quad x_2' = 4x_1 + 3x_2.$$

Solution. First we rewrite the system in the form

$$x' = Ax,$$

where

$$A = \begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A. We set

$$A - \lambda E = \begin{pmatrix} 7-\lambda & -5 \\ 4 & 3-\lambda \end{pmatrix}.$$

Therefore

$$det(A - \lambda E) = \lambda^2 - 10\lambda + 41.$$

Hence we have two eigenvalues

$$\lambda_{1,2} = 5 \pm 4i.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 5 + 4i$. We have

$$A - (5+4i)E = \begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix}.$$

Hence the system $(A - (5+4i)E)\vec{e} = 0$ is equivalent to equation $4e_1 - (2 + 4i)e_2 = 0$. The solution to this equation is $\vec{e} = (1+2i, 2)$. Using this Theorem 2 with $p = 5$ and $q = 4$ we have

$$x_1(t) = e^{5t}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} cos(4t) - \begin{pmatrix} 2 \\ 0 \end{pmatrix} sin(4t)\right),$$

$$x_2(t) = e^{5t}\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} cos(4t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} sin(4t)\right).$$

Then the general solution is

$$X(t) = C_1x_1(t) + C_2x_2(t).$$

Example 4. Solve the initial value problem

$$x_1' = x_1 - 2x_2, \quad x_2' = 2x_1 + x_2, \quad x_1(0) = 0,\ x_2(0) = 4.$$

Solution. First we rewrite the system in the form

$$x' = Ax$$

where

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A. We set

$$A - \lambda E = \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix}.$$

Therefore

$$det(A - \lambda E) = (\lambda - 1)^2 + 4.$$

Hence we have two eigenvalues

$$\lambda_{1,2} = 1 \pm 2i.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 1 + 2i$. We have

$$A - (1 + 2i)E = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}.$$

Hence the system $(A-(1+2i)E)\vec{e} = 0$ is equivalent to equation $2e_1 - 2ie_2 = 0$. The solution to this equation is $\vec{e} = (i, 1)$. Using this Theorem 2 with $p = 1$ and $q = 2$ we have

$$x_1(t) = e^t \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} cos(2t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} sin(2t) \right),$$

$$x_2(t) = e^t \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} cos(2t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} sin(2t) \right).$$

Then the general solution is

$$X(t) = C_1 x_1(t) + C_2 x_2(t).$$

Using the initial conditions we obtain

$$X(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So $C_1 = 4, C_2 = 0$. Therefore the solution to the initial value problem is

$$X(t) = 4e^t \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} cos(2t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} sin(2t) \right).$$