

## LECTURE 34 (267)

Today we consider the first order system of linear differential equations of the form

$$\frac{dx}{dt} = P(t)x + f(t), \quad (1)$$

where  $f(t) = (f_1(t), \dots, f_n(t))$  is given vector function,  $x(t) = (x_1(t), \dots, x_n(t))$  is unknown vector function,  $P(t)$  is the matrix of the size  $n \times n$ . The elements of this matrix are the functions  $p_{ij}(t)$ .

The system (1) with  $f(t) \equiv 0$  is called homogeneous

$$\frac{dx}{dt} = P(t)x, \quad (2)$$

By the Initial Value Problem for the system (1) we mean the following problem: Find a solution to the system (1) a function  $x(t)$  on the interval  $I$  such that  $x(a) = b$ :

$$\frac{dx}{dt} = P(t)x + f(t), \quad x(a) = b. \quad (3)$$

Here  $a$  is a point from the interval  $I$  and  $b = (b_1, \dots, b_n)$  is a given vector.

We have

**Theorem 1.** *Let all elements of the matrix  $P(t)$ , be continuous on the interval  $I$ , and let the function  $f(t)$  be continuous on the interval  $I$ . Assume that  $a \in I$ . Then for each vector  $b$  there exists unique solution  $x(t)$  to the problem (3).*

**Definition.** *The set of functions  $x_1(t), x_2(t), \dots, x_n(t)$  is called linearly dependent if there exists a vector  $(c_1, c_2, \dots, c_n) \neq 0$  such that*

$$c_1x_1(t) + c_2x_2(t) + \dots + c_nx_n(t) = 0$$

*for all  $t \in I$ . Otherwise the set of functions  $x_1(t), x_2(t), \dots, x_n(t)$  is called linearly independent.*

Next we introduce the notion of the Wronskian. We denote the Wronskian as  $W(t)$

$$W(t) = \det(x_1(t), x_2(t), \dots, x_n(t)).$$

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**Theorem 2.** Let all elements of the matrix  $P(t)$ , be continuous on the interval  $I$ , and functions  $x_1(t), x_2(t), \dots, x_n(t)$  be solution to the homogeneous system (2) on the interval  $I$ . Then

**A.** If the functions  $x_1(t), x_2(t), \dots, x_n(t)$  are linearly dependent, then Wronskian of these function  $W(t) = 0$  for all  $t \in I$ .

**B.** If the functions  $x_1(t), x_2(t), \dots, x_n(t)$  are linearly independent, then Wronskian of these function  $W(t) \neq 0$  for all  $t \in I$ .

**Theorem 3.** Let functions  $x_1(t), x_2(t), \dots, x_n(t)$  be solutions to the homogeneous system (2) which are linearly independent on the interval  $I$ . Then the general solution to the system (2) is

$$x_c(t) = C_1x_1(t) + C_2x_2(t) + \dots + C_nx_n(t). \quad (4)$$

**Theorem 4.** Let all elements of the matrix  $P(t)$ , be continuous on the interval  $I$ , and let the function  $f(t)$  be continuous on the interval  $I$ . Then the general solution to the system (1) is

$$x(t) = x_c(t) + x_p(t), \quad (5)$$

where  $x_c(t)$  is the general solution to the homogeneous system given by formula (4), and  $x_p(t)$  is the particular solution to the system (1)

$$\frac{dx_p}{dt} = P(t)x_p + f(t).$$

**Example 1.** Given the system of linear ordinary differential equations

$$x' = y + x + 1, \quad y' = 3x + 3y + \sin(t).$$

Rewrite this system in the form (1).

*Solution.* We set  $\mathbf{x}(t) = (x_1(t), x_2(t))$  and  $x_1(t) = x(t), x_2(t) = y(t)$ . Next we set  $f(t) = (1, \sin(t))$ . Finally we try to determine the matrix  $P(t)$ . We should have

$$P(t) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y + x \\ 3x + 3y \end{pmatrix}.$$

Therefore we have the system

$$p_{11}(t)x + p_{12}(t)y = x + y, \quad p_{21}(t)x + p_{22}(t)y = 3x + 3y.$$

So we have

$$p_{11} = 1, p_{12} = 1, p_{21} = 3, p_{22} = 3.$$

Then

$$P(t) = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$