

LECTURE 34 (267)

Homogeneous systems of linear ordinary differential equations.

Today we consider the homogeneous system of linear ordinary differential equations

$$x' = Ax, \quad (1)$$

where $x(t) = (x_1(t), \dots, x_n(t))$ is a vector function, A is the matrix of the size $n \times n$. The elements of the matrix A are real numbers. Let λ be an eigenvalue of the matrix A and V be an eigenvector associated with this eigenvalue. ($AV = \lambda V$) We have

Theorem 1. *If eigenvalue λ is the real number and V is an eigenvector associated with this eigenvalue then the function*

$$x(t) = e^{\lambda t}V$$

is solution of the system (1).

Next we consider the situation when an eigenvalue λ is complex:

Theorem 2. *If eigenvalue $\lambda = p+iq$ is the complex number and $V = \vec{a}+i\vec{b}$ is an eigenvector associated with this eigenvalue then the functions $x_1(t), x_2(t)$:*

$$x_1(t) = e^{pt}(\vec{a}\cos(qt) - \vec{b}\sin(qt)), \quad x_2(t) = e^{pt}(\vec{b}\cos(qt) + \vec{a}\sin(qt))$$

are solutions to the system (1).

Example 1. Find a general solution to the system

$$x'_1 = x_1 + 2x_2, \quad x'_2 = 2x_1 + x_2.$$

Solution. First we rewrite the system in the form

$$x' = Ax$$

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where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A . We set

$$A - \lambda E = \begin{pmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{pmatrix}$$

Therefore

$$\det(A - \lambda E) = (\lambda - 1)^2 - 4.$$

Hence we have two eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = 3.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = -1$. We have

$$A + E = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

Hence the system $(A + E)\vec{e} = 0$ is equivalent to equation $e_1 + e_2 = 0$. The solution to this equation is $\vec{e} = (1, -1)$.

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 3$. We have

$$A - 3E = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

Hence the system $(A - 3E)\vec{e} = 0$ is equivalent to equation $e_1 - e_2 = 0$. The solution to this equation is $\vec{e} = (1, 1)$. Therefore the general solution to our system is

$$x(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example 2. Find a general solution to the system

$$x'_1 = 2x_1 + 3x_2, \quad x'_2 = 2x_1 + x_2.$$

Solution. First we rewrite the system in the form

$$x' = Ax,$$

where

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A . We set

$$A - \lambda E = \begin{pmatrix} 2 - \lambda & 3 \\ 2 & 1 - \lambda \end{pmatrix}$$

Therefore

$$\det(A - \lambda E) = (2 - \lambda)(1 - \lambda) - 6 = 2 - 2\lambda - \lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda + 4.$$

Hence we have two eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = 4.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = -1$ We have

$$A + E = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}.$$

Hence the system $(A + E)\vec{e} = 0$ is equivalent to equation $e_1 + e_2 = 0$. The solution to this equation is $\vec{e} = (1, -1)$.

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 4$. We have

$$A - 4E = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}.$$

Hence the system $(A - 4E)\vec{e} = 0$ is equivalent to equation $e_1 - 3e_2 = 0$. The solution to this equation is $\vec{e} = (3, 1)$. Therefore the general solution to our system is

$$x(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Example 3. Solve the initial value problem

$$x'_1 = 3x_1 + 4x_2, \quad x'_2 = 3x_1 + 2x_2, \quad x_1(0) = x_2(0) = 1.$$

Solution. First we rewrite the system in the form

$$x' = Ax,$$

where

$$A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}.$$

Next we find an eigenvalues of the matrix A . We set

$$A - \lambda E = \begin{pmatrix} 3 - \lambda & 4 \\ 2 & 2 - \lambda \end{pmatrix}.$$

Therefore

$$\det(A - \lambda E) = (3 - \lambda)(2 - \lambda) - 12 = 6 - 3\lambda - 2\lambda + \lambda^2 - 12 = \lambda^2 - 5\lambda + 6.$$

Hence we have two eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = 6.$$

Lets find eigenvector associated with the eigenvalue $\lambda_1 = -1$. We have

$$A + E = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}.$$

Hence the system $(A + E)\vec{e} = 0$ is equivalent to equation $e_1 + e_2 = 0$. The solution to this equation is $\vec{e} = (1, -1)$.

Lets find eigenvector associated with the eigenvalue $\lambda_1 = 6$. We have

$$A - 6E = \begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix}.$$

Hence the system $(A - 6E)\vec{e} = 0$ is equivalent to equation $3e_1 - 4e_2 = 0$. The solution to this equation is $\vec{e} = (\frac{4}{3}, 1)$. Therefore the general solution to our system is

$$x(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}. \quad (2)$$

We set in (2) $t = 0$. Then

$$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}.$$

Solving the system of the linear equations we obtain $C_2 = \frac{8}{7}$ and $C_1 = \frac{1}{7}$. Hence the solution to the initial value problem is

$$x(t) = \frac{1}{7} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{8}{7} e^{6t} \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}. \quad (2)$$