LECTURE 31

Review of the Matrix Theory.

Example 1. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & 4 \\ 5 & -6 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & -2 & 0 \end{pmatrix}$$

Find $C = AB$ and $\tilde{C} = BA$

Solution. Using the formula for the multiplication of the matrices from the previous lecture we have

$$C = AB = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -3 & -1 \\ 6 & 5 & 5 \end{pmatrix} \quad \tilde{C} = BA = \begin{pmatrix} 4 & -3 & 1 \\ 0 & -1 & 0 \\ 2 & -5 & -2 \end{pmatrix}$$

Let A be a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and β be a number. We introduce the product of the matrix A and the number β by the formula:

$$\beta A = \begin{pmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{pmatrix}$$

Example 2. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & 4 \\ 5 & -6 & 0 \end{pmatrix}$$

and $\beta = 2$. Find βA.

Solution.

$$\beta A = 2\begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & 4 \\ 5 & -6 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 6 & 8 \\ 10 & -12 & 0 \end{pmatrix}$$

Let A and B be two matrices :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

We introduce the sum of these matrices by formula

$$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

Example 3. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & 4 \\ 5 & -6 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & -2 & 0 \end{pmatrix}$$

Find $A + B$.

Solution.

$$A + B = \begin{pmatrix} 0+0 & 1+1 & 0+1 \\ -1-1 & 3+0 & 4+0 \\ 5+1 & -6-2 & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 3 & 4 \\ 6 & -8 & 0 \end{pmatrix}.$$

Definition. *A number λ (real or complex) is called the eigenvalue of the matrix A if there exists a vector $\vec{b} \neq 0$ such that*

$$A\vec{e} = \lambda\vec{e}.$$

Vector $\vec{b}$ is called the eigenvector.

We note that an eigenvector multiplied by nonzero number is an eigenvector also.

Example 4. For the unit matrix E find all eigenvalues and eigenvectors.

Solution. Let $\vec{e} = (e_1, e_2, e_3)$ be a vector such that $\vec{e} \neq 0$. Obviously

$$E\vec{e} = \begin{pmatrix} 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 \\ 0 \cdot e_1 + 2 \cdot e_2 + 0 \cdot e_3 \\ 0 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3 \end{pmatrix} = \vec{e}$$

So by the definition of the eigenvalue $\lambda = 1$ is the eigenvalue of the matrix E and an arbitrary vector $\vec{e}$ such that $\vec{e} \neq 0$ is the eigenvector.

How to find all eigenvalues of the matrix A. The rule is very simple: one should consider the polynomial $det(A - \lambda E)$ respect to the variable λ. The roots of this polynomial are the eigenvalues of the matrix A.

By **diagonal** matrix we mean a matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{1}$$

Example 5. Show that eigenvalues and eigenvectors of the matrix (1) are $\lambda = \lambda_1, \vec{i}; \lambda = \lambda_1, \vec{j}; \lambda = \lambda_1, \vec{k}$.

Solution. The matrix $A - \lambda E$ has the form

$$A - \lambda E = \begin{pmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{pmatrix}$$

Note that

$$det(A - \lambda E) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda).$$

The roots of the polynomial $det(A - \lambda E) = 0$ are $\lambda_1, \lambda_2, \lambda_3$.Let's show that the vector $\vec{i}$ is the eigenvector which corresponds to the eigenvalue $\lambda = \lambda_1$. We have

$$A - \lambda_1 E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 - \lambda_1 & 0 \\ 0 & 0 & \lambda_3 - \lambda_1 \end{pmatrix}$$

Then equation

$$(A - \lambda_1 E)\vec{e} = 0.$$

this equation is equivalent to the system

$$\begin{cases} (\lambda_2 - \lambda_1)e_2 = 0 \\ (\lambda_3 - \lambda_1)e_3 = 0 \end{cases}$$

We set $e_2 = e_3 = 0$ and $e_1 = 1$. Hence $\vec{i}$ is the eigenvector.