LECTURE 3 (267)

We consider the initial value problem

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

$$y(t_0) = y_0 \tag{2}$$

Theorem 1. *Let f, f_y be continuous functions in a disk of radius $\delta > 0$ centered at the point (t_0, y_0). Then there exist $T(\delta)$ such that on the interval $[t_0, t_0 + T(\delta)]$ exist a unique solution to problem (1),(2).*

Let $\bar{y}(t)$ be solution to problem (1) with the initial condition $y(t_0) = y_1$ and $\tilde{y}(t)$ be solution to problem (1) with the initial condition $\tilde{y}(0) = y_2$. Then if $y_1 \neq y_2$ there is no points of intersection between the graphs of the functions $\bar{y}(t)$ and $\tilde{y}(t)$.

Separable equations. We consider equation

$$\frac{dy}{dt} = H(t, y) \tag{3}$$

Suppose that the function $H(t, y)$ can be represented in the form

$$H(t, y) = \frac{g(x)}{f(y)}. \tag{4}$$

In that case solution to the problem (3) satisfies the equation

$$\int f(y)dy = \int g(x)dx + C. \tag{5}$$

Example 1. Find the general solution to ordinary differential equation

$$\frac{dy}{dt} + 3ty = 0$$

Solution. First we reduce the equation to the form (3), (4). We have

$$\frac{dy}{dt} = -3ty = -\frac{-3t}{\frac{1}{y}}.$$

Then $f(y) = \frac{1}{y}$ and $g(t) = -3t$. We have

$$\int g(t)dt = -\frac{3}{2}t^2 \qquad \int \frac{1}{y}dy = \ln|y|.$$

Then by (5)

$$\ln|y| = -\frac{3t^2}{2} + C.$$

And

$$|y(t)| = e^{-\frac{3t^2}{2}} + C.$$

Finally we arrive to

$$y(t) = C_1 e^{-\frac{3t^2}{2}}.$$

Example 2. Solve the initial value problem

$$\frac{dy}{dt} = ty^3, \quad y(0) = 1. \tag{6}$$

Solution. The equation (6) is separable. Really

$$\frac{dy}{dt} = ty^3 = \frac{t}{\frac{1}{y^3}}.$$

So $g(t) = t$ and $f(y) = \frac{1}{y^3}$.

Hence

$$\int g(t)dt = \frac{1}{2}t^2, \quad \int f(y)dy = -\frac{1}{2y^2}.$$

The formula (5) implies

$$-\frac{1}{2y^2(t)} = \frac{1}{2}t^2 + C.$$

Now we fix the constant C:

$$-\frac{1}{2y^2(0)} = -\frac{1}{2} = C$$

Hence $C = \frac{1}{2}$ and

$$-\frac{1}{2y^2(t)} = \frac{1}{2}t^2 + \frac{1}{2}.$$

Then

$$y^2(t) = \frac{1}{1-t^2}$$

This formula implies

$$y(t) = \pm\sqrt{\frac{1}{1-t^2}} \tag{8}$$

We should take in formula (8) the sing + because the initial condition $y(0) = 1$.

Example 3. Solve the initial value problem

$$\frac{dy}{dt} = 6e^{2t-y}, \quad y(0) = 0. \tag{9}$$

Solution We note that equation (9) is separable

$$\frac{dy}{dt} = 6e^{2t-y} = \frac{6e^{2t}}{e^y}.$$

Really

$$g(t) = 6e^{2t} \quad f(y) = e^y.$$

Then

$$\int g(t)dt = \int 6e^{2t}dt = 3e^{2t}, \quad \int e^y dy = e^y.$$

By formula (5) we have

$$e^{y(t)} = 3e^t + C$$

Using the initial condition we can fix the constant C:

$$e^{y(0)} = 1 = 3 + C$$

So $C = -2$ and $e^{y(t)} = 3e^t - 2$. Therefore

$$y(t) = \ln(3e^t - 2).$$