

LECTURE 31

Review of the Matrix Theory.

By the matrix A of the size 2×2 we mean the following object

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}$$

Here the elements of the matrix A - $a_{11}, a_{12}, a_{21}, a_{22}$ are the numbers real or complex.

The matrix A of the size 3×3 is the following object

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The elements of the matrix A are the numbers.

Example.

$$A = \begin{pmatrix} 0 & \frac{3}{2} & 7 \\ -1 & 3 & 4 \\ 5 & -6 & 0 \end{pmatrix}$$

By **determinant** of the matrix A we mean the following number

$$\det A = a_{11}a_{22} - a_{21}a_{12}$$

in the case of the matrix 2×2 and

$$\det A = a_{11}a_{22}a_{33} + a_{31}a_{12}a_{23} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33}$$

in the case of the matrix 3×3 .

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Example 1. Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}$$

Solution.

$$\det A = 1 \cdot 0 - 3(-1) = 3.$$

Example 2. Find the determinant of the matrix

$$A = \begin{pmatrix} 0 & \frac{3}{2} & 7 \\ -1 & 3 & 4 \\ 5 & -6 & 0 \end{pmatrix}$$

Solution.

$$\det A = 0 \cdot 3 \cdot 0 + \frac{3}{2} \cdot 4 \cdot 5 + 7 \cdot (-6) \cdot 1 - 5 \cdot 3 \cdot 7 - 0 \cdot (-6) \cdot 4 - 0 \cdot (-1) \cdot \frac{3}{2} = -117.$$

Let A be a matrix of the size 3×3 and $\vec{b} = (b_1, b_2, b_3)$. We determine the product of the matrix A and the vector \vec{b} :

$$A\vec{b} = (a_{11}b_1 + a_{12}b_2 + a_{13}b_3, a_{21}b_1 + a_{22}b_2 + a_{23}b_3, a_{31}b_1 + a_{32}b_2 + a_{33}b_3).$$

Therefore the product of the matrix A and the vector \vec{b} is the vector.

Example. Find the product $A\vec{b}$

$$A = \begin{pmatrix} 0 & \frac{3}{2} & 7 \\ -1 & 3 & 4 \\ 5 & -6 & 0 \end{pmatrix}, \quad \vec{b} = (1, 0, 1).$$

Solution

$$A\vec{b} = \begin{pmatrix} 0 & \frac{3}{2} & 7 \\ -1 & 3 & 4 \\ 5 & -6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + \frac{3}{2} \cdot 0 + 7 \cdot 1 \\ -1 \cdot 1 + 3 \cdot 0 + 4 \cdot 1 \\ 5 \cdot 1 + (-6) \cdot 0 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix}.$$

Let $Y = (y_1, y_2, y_3)$ be the given vector and $X = (x_1, x_2, x_3)$ be unknown vector. We consider the equation

$$AX = Y. \tag{1}$$

Here A is a matrix. We observe that equation (1) is the compact form of writing for the linear system of three equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = y_3 \end{cases} \tag{2}$$

We have

Theorem 1. *If $\det A \neq 0$ then the system (1) has unique solution X for each vector Y . If $\det A = 0$ there exists vector \tilde{Y} such that there is no solution to equation*

$$AX = \tilde{Y}.$$

Moreover there exists $\tilde{X} \neq 0$ such that $A\tilde{X} = 0$.

Next we introduce the product of two matrices A and B :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

By the product of the matrices A and B we mean the matrix C

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = AB,$$

such that the elements of the matrix C are given by formula

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}.$$

We note that in general

$$AB \neq BA.$$