LECTURE 29 (267)

Periodic and Piecewise continuous functions.

We consider the spring-mass -dashpot system with the external force $f(t)$:

$$mx'' + cx' + kx = f(t) \quad x(0) = x'(0) = 0. \tag{1}$$

Example 1. Let $m = 1, k = 4, c = 4, f(t) = 1 \quad \text{if } 0 \le t < 2, \quad f(t) = 0$ if $t \ge 2$. Find $x(t)$.

Solution. In our particular case equation (1) has the form

$$x'' + 4x' + 4x = f(t)$$

Taking the Laplace transform of this ordinary differential equation we obtain

$$s^2F(s) + 4sF(s) + 4F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{1 - u(t-2)\} = \frac{1}{s} - \frac{e^{-2s}}{s}.$$

Therefore we have

$$F(s) = \frac{1}{s}\frac{1}{(s+2)^2} - e^{-2s}\frac{1}{s}\frac{1}{(s+2)^2}$$

Note that

$$\mathcal{L}^{-1}\{\frac{1}{s}\frac{1}{(s+2)^2}\} = \int_0^t \tau e^{-2\tau} d\tau = -\frac{1}{2}(\tau - \frac{1}{2})e^{-2\tau}|_0^t = -\frac{1}{2}(t - \frac{1}{2})e^{-2t} - \frac{1}{4}.$$

Finally we have

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{2}(t - \frac{1}{2})e^{-2t} - \frac{1}{4}. + u(t-2)(\frac{1}{2}(t - 2 - \frac{1}{2})e^{-2(t-2)} - \frac{1}{4}).$$

Example 2. We consider RLC circuit described by the the ordinary differential equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau = e(t), i(0) = 0. \tag{2}$$

Let $L = 1, R = 0, C = 10^{-4}, e(t) = 100$ if $0 \leq t < 2\pi; e(t) = 0$ if $t \geq 2\pi$.

Solution. In our case equation (2) has the form

$$\frac{di}{dt} + 10^4 \int_0^t i(\tau)d\tau = e(t), i(0) = 0. \qquad (3)$$

Applying the Laplace transform to the right and left hand sides of equation (3) we obtain

$$sF(s) - i(0) + 10^4 \frac{F(s)}{s} = \mathcal{L}\{e\} = \mathcal{L}\{tu(t-2)\} = \mathcal{L}\{(t-2)u(t-2)\} + 2\mathcal{L}\{u(t-2)\}$$

Since $i(0) = 0$ we obtain

$$sF(s) + 10^4 \frac{F(s)}{s} = e^{-2s}\frac{1}{s^2} + 2e^{-2s}\frac{1}{s}.$$

Multiplying this equation by s we have

$$s^2F(s) + 10^4F(s) = e^{-2s}\frac{1}{s} + 2e^{-2s}.$$

Therefore

$$F(s) = \frac{e^{-2s}\frac{1}{s} + 2e^{-2s}}{s^2 + 10^4}.$$

Note that

$$\mathcal{L}^{-1}\{\frac{2e^{-2s}}{s^2 + 10^4}\} = \frac{1}{50}u(t-2)sin(100(t-2)) \qquad (4)$$

and

$$\frac{1}{s(s^2 + 10^4)} = \frac{1}{s}\frac{1}{s^2 + 10^4} = \mathcal{L}\{1\}\mathcal{L}\{\frac{1}{100}sin(100t)\}$$

So by the convolution theorem

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2 + 10^4)}\} = \frac{1}{100}\int_0^t sin(100\tau)d\tau = -\frac{1}{100^2}cos(100\tau)|_0^t = \frac{1}{100^2} - \frac{1}{100^2}cos(100t).$$

Therefore we have

$$\mathcal{L}^{-1}\{\frac{e^{-2s}}{s(s^2 + 10^2)}\} = u(t-2)(\frac{1}{100^2} - \frac{1}{100^2}cos(100(t-2)). \qquad (5)$$

From (4),(5) we finally obtain

$$i(t) = \frac{1}{50}u(t-2)sin(100(t-2)) + u(t-2)(\frac{1}{100^2} - \frac{1}{100^2}cos(100(t-2)).$$

Definition 1. *The function $f(t)$ defined for $t \geq 0$ is said to be periodic function with the period $p > 0$ if*

$$f(t+p) = f(t)$$

for all $t \geq 0$.

For the Laplace transform of the periodic function we have the following formula

Theorem. *Let $f(t)$ be periodic function with period p and piecewise continuous for $t \geq 0$. Then the transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > 0$ and is given by*

$$F(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t) dt. \tag{6}$$

Example 3. We consider the spring-mass dashpot system with $m = 1, c = 0, k = 4$ and $f(t)$ is the periodic function with the period 2π such that $f(t) = 10$ if $0 \leq t \leq \pi$ and $f(t) = -10$ if $\pi < t < 2\pi$.

Solution. For our parameters the ordinary differential equation which describe our system is

$$x'' + 4x = f(t), \quad x(0) = 0, x'(0) = 0. \tag{7}$$

Applying the Laplace transform to equation (6) we obtain

$$s^2 F(s) + 4F(s) = \mathcal{L}\{f(t)\}.$$

Applying the formula (6) we obtain

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}} (\int_0^\pi e^{-st} 10 dt - 10 \int_\pi^{2\pi} e^{-st} dt) =$$

$$\frac{1}{1-e^{-2\pi s}} (10\frac{1}{s} - 20e^{-\pi s}\frac{1}{s} + 10e^{-2\pi s}\frac{1}{s}) = 10(1-e^{-2\pi s}).$$

Hence

$$F(s) = \frac{10(1-e^{-2\pi s})}{s^2+4}$$

Finally

$$x(t) = \mathcal{L}^{-1}\{F(s)\} = 5sin(2t) - 5u(t-2\pi)sin(2(t-2\pi)).$$