LECTURE 28 (267)

Periodic and Piecewise continuous functions.

First we remind the definition of the step function

$$u_a(t) = u(t-a) = \begin{cases} 0 & \text{if } \ t \le a, \\ 1 & \text{if } t \ge a. \end{cases}$$

Theorem 1. *(Translation on the t-Axis) If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then*

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \tag{1}$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a). \tag{2}$$

Example 1. Find inverse Laplace transform for the function

$$G(s) = \frac{e^{-3s}}{s^2}$$

Solution. We know that

$$f(t) = \mathcal{L}\{t\} = \frac{1}{s^2}.$$

Hence using the formula (1) with $F(s) = \frac{1}{s^2}$ and $a = 3$ we obtain

$$\mathcal{L}^{-1}\{\frac{e^{-3s}}{s^2}\} = u(t-3)(t-3).$$

Example 2. Find inverse Laplace transform for the function

$$G(s) = \frac{e^{-s} - e^{-3s}}{s^2}$$

Solution. We know that

$$f(t) = \mathcal{L}\{t\} = \frac{1}{s^2}.$$

Hence using the formula (1) with $F(s) = \frac{1}{s^2}$ and $a = 1, 3$ we obtain

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{e^{-s} - e^{-3s}}{s^2}\} = \mathcal{L}^{-1}\{\frac{e^{-s}}{s^2}\} - \mathcal{L}^{-1}\{\frac{e^{-3s}}{s^2}\} = u(t-1)(t-1) - u(t-3)(t-3)$$

Example 3. Find inverse Laplace transform for the function

$$G(s) = s\frac{1 + e^{-3s}}{s^2 + \pi^2}$$

Solution. First we observe

$$f(t) = \mathcal{L}^{-1}\{\frac{s}{s^2 + \pi^2}\} = cos(\pi t)$$

We know that

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{s}{s^2 + \pi^2}\} + \mathcal{L}^{-1}\{\frac{se^{-3s}}{s^2 + \pi^2}\} = cos(\pi t) + u(t-3)cos(\pi(t-3)).$$

Example 4. Find the Laplace transform of the function

$$f(t) = 2 \quad \text{if } 0 \leq t < 3; \; f(t) = 0 \quad \text{if} \, t \geq 3.$$

Solution. We represent the function $f(t)$ in the form

$$f(t) = 2(1 - u(t-3)).$$

Hence

$$\mathcal{L}\{f(t)\} = 2(\frac{1}{s} - e^{-3s}).$$

Example 5. Find the Laplace transform of the function

$$f(t) = cos(\pi t) \quad \text{if } 0 \leq t < 2; \; f(t) = 0 \quad \text{if } t \geq 2.$$

Solution. We represent the function $f(t)$ in the form

$$f(t) = cos(\pi t)(1 - u(t-2)) = cos(\pi t) - cos(\pi(t-2))u(t-2).$$

Hence

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+\pi^2} - e^{-2s}\frac{s}{s^2+\pi^2}.$$

Example 6. Find the Laplace transform of the function

$$f(t) = t^3 \quad \text{if } 1 \le t < 2; \;\; f(t) = 0 \quad \text{if } t \ge 2 \quad \text{or } 0 < t < 1.$$

Solution. We represent the function $f(t)$ in the form

$$f(t) = t^3(u(t-1) - u(t-2))$$

Therefore

$$\mathcal{L}\{f(t)\} = \frac{d^3}{ds^3}(\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}).$$

We consider the spring-mass -dashpot system with the external force $f(t)$:

$$mx'' + cx' + kx = f(t) \quad x(0) = x'(0) = 0. \tag{3}$$

Example 7. Let $m = 1, k = 4, c = 0, f(t) = 1 \quad \text{if } 0 \le t < \pi, f(t) = 0$ if $t \ge \pi$. Find $x(t)$.

Solution. In our particular case equation (3) has the form

$$x'' + 4x = f(t)$$

Taking the Laplace transform of this ordinary differential equation we obtain

$$s^2F(s) + 4F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{1 - u(t-\pi)\} = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

Therefore we have

$$F(s) = \frac{1}{s}\frac{1}{s^2+2^2} - e^{-\pi s}\frac{1}{s}\frac{1}{s^2+2^2}$$

Note that

$$\mathcal{L}^{-1}\{\frac{1}{s}\frac{1}{s^2+2^2}\} = \frac{1}{2}\int_0^t sin(2\tau)d\tau = -\frac{cos(2\tau)}{4}|_0^t = \frac{1}{4} - \frac{cos(2t)}{4}$$

Finally we have

$$x(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{4} - \frac{cos(2t)}{4} - u(t-\pi)(\frac{1}{4} - \frac{cos(2(t-\pi))}{4}).$$