APPLICATIONS OF TAYLOR POLYNOMIALS, BINOMIAL SERIES.

We start from some important formulas

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

Here

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}, \qquad \binom{k}{0} = 1.$$

In general, let a and b be two numbers

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n.$$

Theorem.(BINOMIAL SERIES). *If k is my real number and $|x| < 1$ then*

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \tag{1}$$

Example 1. Find Maclaurin series for the function $f(x) = (1+x^2)^{\frac{1}{3}}$.
Solution. We use the formula (1) with $k = \frac{1}{3}$.

$$(1+y)^{\frac{1}{3}} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} y^n. \tag{2}$$

In formula (2) we plug in instead of y the x^2

$$(1+x^2)^{\frac{1}{3}} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} x^{2n}.$$

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Example 2. Find Maclaurin series for the function $f(x) = \frac{x^2}{\sqrt{(2+x)}}$.

*Solution.*First we transform the function $f(x)$ to more convinient form

$$f(x) = \frac{x^2}{\sqrt{(2+x)}} = \frac{x^2}{\sqrt{2}}\frac{1}{\sqrt{1+\frac{x}{2}}}.$$

We set $y = \frac{x}{2}$. Using the formula (1) we have

$$\frac{1}{\sqrt{1+\frac{x}{2}}} = \frac{1}{\sqrt{1+y}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n} y^n = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{1}{2^n}x^n. \tag{3}$$

Using (3) we have

$$f(x) = \frac{x^2}{\sqrt{2}}\frac{1}{\sqrt{1+\frac{x}{2}}} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{1}{2^n}x^n = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{1}{\sqrt{2}2^n}x^{n+2}.$$

Example 3. For the function $f(x) = \frac{1}{\sqrt{1+x^2}}$ evaluate $f^{(10)}(0)$.

Solution. First we find the representation of the function $f(x)$ in the form of the Maclaurin series

We use the formula (1) with $k = -\frac{1}{2}$.

$$(1+y)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n} y^n.$$

In formula (2) we plug in instead of y the x^2

$$(1+x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n} x^{2n}. \tag{4}$$

Next we note if $f(x) = \sum_{n=0}^{\infty} c_n x^n$ then $c_n = \frac{f^{(n)}(0)}{n!}$. Hence

$$f^{(n)}(0) = c_n n!.$$

From (4) we have $c_{10} = \binom{-\frac{1}{2}}{10}$. Therefore

$$f^{(10)}(0) = \binom{-\frac{1}{2}}{10} 10!.$$

Example 4. Determine the number of terms in Maclaurin series for e^x that should be used to estimate $e^{0,1}$ with an error less then 0.001.

Solution. We have

$$|f(x) - T_n(x)| \leq |R_n(x)| \quad \text{for all } x \in (-a, a),$$

where

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for all } x \in (-a, a),$$

and

$$M = sup_{x\in(-a,a)}|f^{(n+1)}(x)|.$$

We take $a = 0.1$. Then $M = sup_{x\in(-a,a)}|f^{(n+1)}(x)| = M = sup_{x\in(-0.1,0.1)}|e^x| = e^{0.1}$. Therefore

$$|R_n(x)| \leq \frac{e^{0.1}}{(n+1)!}0.1^{n+1} \quad \text{for all } x \in (-0.1, 0.1).$$

If we take $n = 2$ then

$$|R_n(x)| \leq \frac{e^{0.1}}{(n+1)!}0.1^{n+1} \leq \frac{e^{0.1}}{3!}0.001 \leq 0.001 \quad \text{for all } x \in (-0.1, 0.1).$$