LECTURE 27 (267)

Derivatives, Integrals and Products of Transforms.

Example 1. Find the inverse Laplace transform for the function

$$G(s) = \ln(1 + \frac{1}{s^2}) = \ln(\frac{1+s^2}{s^2}).$$

Solution. We transform our function $G(s)$ as

$$G(s) = \ln(\frac{1+s^2}{s^2}) = -\ln\frac{s^2}{1+s^2} = -\ln\frac{s^2}{1+s^2} =$$

$$\lim_{N\to+\infty}\int_s^N (\frac{2}{\sigma} - \frac{2\sigma}{1+\sigma^2})d\sigma = \int_s^{+\infty} (\frac{2}{\sigma} - \frac{2\sigma}{1+\sigma^2})d\sigma$$

Next we set

$$F(s) = \frac{2}{s} - \frac{2s}{1+s^2}.$$

We observe that

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = 2 - 2cos(t)$$

Therefore

$$\mathcal{L}^{-1}\{G(s)\} = \frac{f(t)}{t} = \frac{2-2cos(t)}{t}.$$

Example 2. Find the inverse Laplace transform for the function

$$G(s) = \frac{s}{(s^2+1)^3}.$$

Solution. Note that

$$G(s) = \frac{s}{(s^2+1)^3} = -\frac{d}{ds}\frac{1}{4}\frac{1}{(s^2+1)^2}$$

Since

$$\mathcal{L}^{-1}\{\frac{1}{4}\frac{1}{(s^2+1)^2}\} = \frac{1}{8}(sin(t) - tcos(t))$$

We have

$$\mathcal{L}^{-1}\{G(s)\} = \frac{t}{8}(sin(t) - tcos(t))$$

Example 3. Find nontrivial solution to the ordinary differential equation

$$tx'' + (t-2)x' + x = 0 \tag{1}$$

such that $x(0) = 0$

Solution. We apply the Laplace transform to the both parts of the linear ordinary differential equation (1)

$$\mathcal{L}\{tx'' + (t-2)x' + x\} = -\frac{d}{ds}\mathcal{L}\{x''\} - \frac{d}{ds}\mathcal{L}\{x'\} - 2\mathcal{L}\{x'\} + \mathcal{L}\{x\}$$

$$-\frac{d}{ds}(s^2F(s) - sx(0) - x'(0)) - \frac{d}{ds}(sF(s) - x(0)) - 2sF(s) + F(s) = 0.$$

Since $x(0) = 0$ we have

$$-\frac{d}{ds}(s^2F(s) - x'(0)) - \frac{d}{ds}(sF(s)) - 2sF(s) + F(s) = 0.$$

or

$$-s^2F'(s) - sF'(s) - 4sF(s) = 0$$

Dividing by $-s$ both parts of this equation we obtain

$$(s+1)F'(s) + 4F(s) = 0.$$

or

$$F'(s) + \frac{4}{(s+1)}F(s) = 0$$

The solution to this first order ordinary differential equation is

$$F(s) = e^{-\int \frac{4}{s+1}ds} = \frac{1}{(1+s)^4}.$$

Finally

$$x(t) = \mathcal{L}\{F(s)\} = \frac{1}{6}e^{-t}t^3.$$

Example 4. Find nontrivial solution to the ordinary differential equation

$$tx'' - 2x' + tx = 0 \tag{2}$$

such that $x(0) = 0$

Solution. We apply the Laplace transform to the both parts of the linear ordinary differential equation (1)

$$\mathcal{L}\{tx'' - 2x' + tx\} = -\frac{d}{ds}\mathcal{L}\{x''\} - 2\mathcal{L}\{x'\} - \frac{d}{ds}\mathcal{L}\{x\}$$

$$-\frac{d}{ds}(s^2F(s) - sx(0) - x'(0)) - 2sF(s) + 2x(0) - F'(s) = 0.$$

Since $x(0) = 0$ we have

$$-\frac{d}{ds}(s^2F(s) - x'(0)) - 2sF(s) - F'(s) = 0.$$

and

$$-(s^2+1)F'(s) - 4sF(s) = 0$$

Hence

$$F'(s) + \frac{4s}{s^2+1}F(s) = 0$$

The solution to this linear O.D.E. is

$$F(s) = e^{-\int \frac{4s}{s^2+1}ds} = e^{-2\ln(1+s^2)} = \frac{1}{(1+s^2)^2}$$

Finally

$$x(t) = \mathcal{L}^{-1}\{\frac{1}{(1+s^2)^2}\} = \frac{1}{2}(sin(t) - tcos(t))$$

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Example 5.

$$x'' + 4x' + 13x = f(t), \quad x(t) = \frac{1}{3}\int_0^t f(t-\tau)e^{-2\tau}sin(3\tau)d\tau.$$

Solution. Taking the Laplace transform of the ordinary differential equation we have

$$s^2F(s) + 4sF(s) + 13F(s) = \mathcal{L}\{f\}$$

Then

$$F(s) = \frac{\mathcal{L}\{f\}}{s^2+4s+13} = \frac{\mathcal{L}\{f\}}{(s+2)^2+3^2} = \mathcal{L}\{f\}\mathcal{L}\{\frac{1}{3}e^{-2t}sin(3t)\}.$$

Applying the convolution formula from the previous lecture we obtain

$$x(t) = \frac{1}{3}\int_0^t f(t-\tau)e^{-2\tau}sin(3\tau)d\tau.$$