REVIEW FOR THE CHAPTER 10

Problem 1. Find all x such that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}} 3^n (x-1)^n$ is convergent.

Solution. We use the Ratio Test.

$$a_n = \frac{3^n}{n^{\frac{1}{3}}}(x-1)^n, \quad a_{n+1} = \frac{3^{n+1}}{(n+1)^{\frac{1}{3}}}(x-1)^{n+1}.$$

Then

$$|\frac{a_{n+1}}{a_n}| = \frac{3^{n+1}|x-1|^{n+1} n^{\frac{1}{3}}}{(n+1)^{\frac{1}{3}} 3^n |x-1|^n} = (\frac{n}{n+1})^{\frac{1}{3}} 3|x-1|.$$

Hence

$$\lim_{n \to +\infty} |\frac{a_{n+1}}{a_n}| = 3|x-1|.$$

By the Ratio Test we have: The series is convergent if $3|x-1| < 1$ and the series is divergent if $3|x-1| > 1$. Therefore for $|x-1| < \frac{1}{3}$ the infinite series is convergent and for $|x-1| > \frac{1}{3}$ the infinite series is divergent. For $x = \frac{4}{3}$ our infinite series has the form $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ this is the p-series with $p = \frac{1}{3}$. So it is divergent.

For $x = \frac{2}{3}$ our infinite series has the form $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{3}}}$ We use the Alternating Series Test. Here $b_n = n^{\frac{1}{3}}$ are positive. Also the sequence $\{b_n\}$ is decreasing. Hence the infinite series is Convergent.

Problem 2 Determine the convergence and absolute convergence of the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1-cos(\frac{1}{n})}{n}$.

Solution. We check the absolute convergence first. By the definition of the absolute convergence we should consider the infinite series $\sum_{n=1}^{\infty} \frac{|1-cos(\frac{1}{n})|}{n}$ Let us compute the limit

$$\lim_{x \to +0} \frac{1-cos(x)}{x^2} = \lim_{x \to +0} \frac{sin(x)}{2x} = \lim_{x \to +0} \frac{cos(x)}{2} = \frac{1}{2}.$$

Hence for all sufficiently small x we have

$$|1 - cos(x)| \leq x^2$$

Therefore

$$|1 - cos(\frac{1}{n})| \leq \frac{1}{n^2}$$

for all sufficiently large n. This inequality imply

$$\frac{|1 - cos(\frac{1}{n})|}{n} \leq \frac{1}{n^3}$$

for all sufficiently large n. Since the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, our infinite series is convergent by the Comparison Test.

Problem 3 Find the representation for the function $f(x) = \ln(1 + 2x) + \frac{x}{1+x}$ in the form of Maclaurin series.

Solution First we find the Maclaurin series for the function $\frac{x}{1+x}$

$$\frac{x}{1+x} = 1 - \frac{1}{1+x} = 1 - \sum_{n=0}^{\infty}(-1)^n x^n = -\sum_{n=1}^{\infty}(-1)^n x^n = \sum_{n=1}^{\infty}(-1)^{n+1} x^n = \sum_{n=0}^{\infty}(-1)^n x^{n+1}$$

For the function $\ln(1 + 2x)$ we have

$$\ln(1+2x) = \frac{1}{2}\frac{d}{dx}\int \frac{1}{1+2x}dx = \frac{1}{2}\int \sum_{n=0}^{\infty}(-1)^n 2^n x^n dx = \int \sum_{n=0}^{\infty}(-1)^n 2^{n-1} x^n dx = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n-1} x^{n+}}{n+1}$$

Finally

$$\ln(1+2x) + \frac{x}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n-1} x^{n+1}}{n+1} + \sum_{n=0}^{\infty}(-1)^n x^{n+1} = \sum_{n=0}^{\infty}(\frac{(-1)^n 2^{n-1}}{n+1} + (-1)^n)x^n.$$

Problem 4. Find the antiderivative $\int \frac{cos(2t)-1}{t^2} dt$

Solution. For the function $cos(2t)$ we have

$$cos(2t) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{2n!}.$$

Therefore

$$\int \frac{cos(2t) - 1}{t^2} dx = \int \frac{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{2n!} - 1}{x^2} dx = \int \frac{\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{2n!}}{x^2} dx =$$

$$\int \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n-2}}{2n!} dx = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n-1}}{2n!(2n-1)}$$