

## LECTURE 26 (267)

Derivatives, Integrals and Products of Transforms.

**Definition.** The convolution  $f * g$  of the piecewise functions  $f$  and  $g$  is defined for  $t \geq 0$  as follows:

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau. \quad (1)$$

We have the following important Theorem:

**Theorem 1.** (Convolution property) Suppose that  $f(t)$  and  $g(t)$  are piecewise continuous for  $t \geq 0$  and that  $|f(t)|$  and  $|g(t)|$  are bounded by  $Me^{ct}$  as  $t \rightarrow +\infty$ . Then the Laplace transform on the convolution  $f(t) * g(t)$  exists for  $s > c$ ; moreover

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\} \quad (2)$$

and

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t). \quad (3)$$

**Theorem 2.** (Differentiation of Transforms) If  $f(t)$  is piecewise continuous for  $t \geq 0$  and  $|f(t)| \leq Me^{ct}$  as  $t \rightarrow +\infty$ , then

$$\mathcal{L}\{-tf(t)\} = F'(s) \quad (4)$$

for  $s > c$ . Equivalently

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}. \quad (5)$$

Also we have

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s). \quad (6)$$

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**Theorem 3.** Suppose that  $f(t)$  is piecewise continuous for  $t \geq 0$ , that  $f(t)$  satisfies the following condition

$$\lim_{t \rightarrow +0} \left\{ \frac{f(t)}{t} \right\} \text{ exists and is finite.}$$

Then

$$\mathcal{L} \left\{ \frac{f(t)}{t} \right\} = \int_s^\infty F(\sigma) d\sigma \quad (7)$$

for  $s > c$ .

**Example 1.** Find the inverse Laplace transform for the following function

$$F(s) = \frac{1}{s(s-3)}.$$

*Solution.* We are going to use the convolution property

$$F(s) = \frac{1}{s(s-3)} = \mathcal{L}\{1\} \mathcal{L}\{e^{3t}\} = \int_0^t e^{3\tau} d\tau = \frac{1}{3} e^{3\tau} \Big|_0^t = \frac{1}{3} (e^{3t} - 1).$$

**Example 2.** Find the inverse Laplace transform for the following function

$$F(s) = \frac{s^2}{(s^2 + 4)^2}.$$

*Solution.* We are going to use the convolution property

$$\begin{aligned} F(s) &= \frac{s^2}{(s^2 + 4)^2} = \frac{s}{s^2 + 4} \frac{s}{s^2 + 4} = \mathcal{L}\{\cos(2t)\} \mathcal{L}\{\cos(2t)\} = \\ &= \int_0^t \cos(2\tau) \cos(2t - 2\tau) d\tau = \frac{1}{2} \int_0^t \cos(2t) + \cos(4\tau - 2t) dt = \\ &= \frac{t}{2} \cos(2t) + \frac{1}{8} \sin(4\tau - 2t) \Big|_0^t = \frac{t}{2} \cos(2t) + \frac{\sin(2t)}{4}. \end{aligned}$$

**Example 3.** Find the inverse Laplace transform for the following function

$$F(s) = \frac{s}{(s^2 + 1)(s - 3)}.$$

*Solution.* We are going to use the convolution property

$$F(s) = \frac{s}{(s^2 + 1)(s - 3)} = \mathcal{L}\{\cos(t)\}\mathcal{L}\{e^{3t}\} = \int_0^t \cos(\tau)e^{3(t-\tau)}d\tau = e^{3t} \int_0^t \cos(\tau)e^{-3\tau}d\tau = e^{3t} \frac{e^{-3\tau}}{10} (3\cos(\tau) + \sin(\tau)) \Big|_0^t = \frac{1}{10} (3\cos(t) + \sin(t) - 3e^{3t}).$$

**Example 4.** Find the Laplace transform to the function  $t\sin(t)$

*Solution.* We are going to use the Theorem 2. Let  $f(t) = \sin(t)$ . Then  $F(s) = \frac{1}{s^2+1}$ . By the formula (4) we have

$$\mathcal{L}\{t\sin(t)\} = -\frac{d}{ds} \left( \frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2}.$$

**Example 5.** Find the Laplace transform to the function  $t\sin(t)$

*Solution.* We are going to use the Theorem 2. Let  $f(t) = e^{3t}\cos(2t)$ . Then  $F(s) = \frac{s-3}{(s-3)^2+1}$ . By the formula (4) we have

$$\mathcal{L}\{t\sin(t)\} = -\frac{d}{ds} \left( \frac{s-3}{(s-3)^2+1} \right) = -\frac{1}{(s-3)^2+1} + \frac{2(s-3)^2}{((s-3)^2+1)^2}.$$

**Example 6.** Find the inverse Laplace transform for the function

$$G(s) = \ln \frac{s-2}{s+2}.$$

*Solution.* In order to find the inverse Laplace transform we are going to use the formula (6): Really

$$G(s) = \ln \frac{s-2}{s+2} = \ln(s-2) - \ln(s+2) = -\lim_{N \rightarrow +\infty} \ln \left( \frac{\sigma-2}{\sigma+2} \right) \Big|_s^N = -\lim_{N \rightarrow +\infty} \int_s^N \left( \frac{1}{\sigma-2} - \frac{1}{\sigma+2} \right) d\sigma = \int_s^\infty \left( \frac{1}{\sigma+2} - \frac{1}{\sigma-2} \right) d\sigma.$$

We set

$$F(s) = \frac{1}{s+2} - \frac{1}{s-2}.$$

We observe that for the function  $f(t) = e^{-2t} - e^{2t}$  we have

$$\mathcal{L}\{f\} = \left\{ \frac{1}{s+2} - \frac{1}{s-2} \right\} = F(s).$$

Thus applying the formula (7) we have

$$\mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \mathcal{L}\left\{ \frac{e^{-2t} - e^{2t}}{t} \right\} = \ln \frac{s-2}{s+2}.$$