TAYLOR AND MACLAURIN SERIES.

Definition. *If f has a power series representation at a that is, if*

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{for } \ |x-a|<R$$

then its coefficient are given by formula

$$c_n=\frac{f^{(n)}(a)}{n!}.$$

Definition. *The power series of the form*

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the Taylor Series.

Definition. *The particular case of the Taylor series for $a=0$ is called the Maclaurin series.*

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Theorem. *If $f(x)=T_n(x)+R_n(x)$ where T_n is the nth-degree Taylor polynomial of f at a and*

$$\lim_{n\to\infty} R_n(x)=0$$

for $|x-a|<R$ then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

Theorem. *(Taylors inequality) If $|f^{(n+1)}(x)| \leq M$ for all $|x-a|R$ then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$$|R_n(x)\| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad for \;\; |x-a| < R \tag{1}$$

Example 1. Find the Maclaurin series for the function $f(x) = sin^2(x)$.
Solution. Note that $sin^2(x) = \frac{1}{2}(1 - cos(2x))$. Hence

$$sin^2(x) = \frac{1}{2}(1-cos(2x)) = \frac{1}{2}(1-\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{2n!}) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{2n!} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{2n!}$$

Example 2. Evaluate the indefinite integral $\int sin(x^2)dx$ as infinite series.
Solution. Note that

$$\int sin(x^2)dx = \int \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n}}{2n!}dx = C + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{2n-1}x^{2n+1}}{2n!(2n+1)}$$

Example 3. Find the Maclaurin series for the function $f(x) = cos(x^3)$.
*Solution.*We set $y = x^3$. Then $f(x) = cos(x^3) = cos(y) = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{2n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n}}{2n!}$.

Example 4. Find the Maclaurin series for the function $f(x) = xe^{-x}$.
Solution. Note that $e^{-x} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ Therefore

$$xe^{-x} = x\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n!}.$$

Example 5. Find the limit $\lim_{x\to 0}\frac{1-cos(x)}{1+x-e^x}$.
Solution. We decompose the functions $cos(x)$ and e^x using the Taylor's polynomials:

$$cos(x) = 1 - \frac{x^2}{2} + R_3(x), \quad e^x = 1 + x + \frac{x^2}{2} + \tilde{R}_2(x).$$

By formula (1) we have the estimates

$$|R_3(x)| \leq \frac{x^4}{6}, |\tilde{R}_3(x)| \leq \frac{e}{2}|x|^3 \; x \in [-1,1].$$

Hence

$$\lim_{x\to 0}\frac{1-cos(x)}{1+x-e^x}=\lim_{x\to 0}\frac{\frac{x^2}{2}+R_3(x)}{-\frac{x^2}{2}+\tilde{R}_2(x)}=-1.$$

Example 6. Find the limit $\lim_{x\to 0}\frac{sin(x)-x+\frac{x^3}{6}}{x^5}$.

Solution.

$$sin(x)=x-\frac{x^3}{6}+\frac{x^5}{5!}+R_6(x),$$

Therefore

$$\lim_{x\to 0}\frac{sin(x)-x+\frac{x^3}{6}}{x^5}=\lim_{x\to 0}\frac{x-\frac{x^3}{6}+\frac{x^5}{5!}+R_6(x)-x+\frac{x^3}{6}}{x^5}=\lim_{x\to 0}\frac{\frac{x^5}{5!}+R_6(x)}{x^5}=\frac{1}{5!}.$$