LECTURE 25 (267)

Translation and Partial Fractions.

Example 1.

$$x^{(4)} + x = 0, \quad x(0) = 1, x'(0) = x''(0) = x^{(3)}(0) = 0.$$

Solution. Taking the Laplace transform of the ordinary differential equation we have

$$s^4 F(s) - s^3 + F(s) = 0.$$

So

$$F(s) = \frac{s^3}{s^4 + 1}.$$

We need to find the inverse Laplace transform for the function $F(s)$. The roots of the polynomial $Q(s) = s^4 + 1$ are $z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \bar{z}_1 = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$, $z_2 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \bar{z}_2 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. We can factorize the polynomial $s^4 + 1$ as

$$s^4 + 1 = (s^2 - \sqrt{2}s + 1)(s^2 + \sqrt{2}s + 1).$$

So according to the Rule 2 we have to look for the decomposition of the function $F(s)$ in the form

$$F(s) = \frac{As + B}{(s^2 - \sqrt{2}s + 1)} + \frac{Cs + D}{(s^2 + \sqrt{2}s + 1)} =$$
$$\frac{As^3 + Bs^2 + \sqrt{2}As^2 + \sqrt{2}Bs + As + B + Cs^3 + Ds^2 - \sqrt{2}Cs^2 - \sqrt{2}Ds + Cs + D}{s^4 + 1}.$$

Now we set up the system of linear equations

$$\begin{cases} A + C = 1 \\ B + \sqrt{2}A + D - \sqrt{2}C = 0 \\ B + D = 0 \\ A + \sqrt{2}B - \sqrt{2}D + C = 0 \end{cases} \begin{cases} A + C = 1 \\ \sqrt{2}A - \sqrt{2}C = 0 \\ B + D = 0 \\ \sqrt{2}B - \sqrt{2}D + 1 = 0 \end{cases} \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2} \\ B + D = 0 \\ \sqrt{2}B - \sqrt{2}D + 1 = 0 \end{cases}$$

Finally we have

$$\begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2} \\ B = -\frac{1}{2\sqrt{2}} \\ D = +\frac{1}{2\sqrt{2}} \end{cases}$$

Then

$$F(s) = \frac{1}{2}\frac{s - \frac{1}{\sqrt{2}}}{(s^2 - \sqrt{2}s + 1)} + \frac{1}{2}\frac{s + \frac{1}{\sqrt{2}}}{(s^2 + \sqrt{2}s + 1)} = \frac{1}{2}\frac{s - \frac{1}{\sqrt{2}}}{(s - \frac{1}{\sqrt{2}})^2 + \frac{1}{2}} + \frac{1}{2}\frac{s + \frac{1}{\sqrt{2}}}{(s - \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}.$$

Therefore

$$x(t) = \frac{1}{2}e^{\frac{t}{\sqrt{2}}}cos(\frac{t}{\sqrt{2}}) + \frac{1}{2}e^{\frac{-t}{\sqrt{2}}}cos(\frac{t}{\sqrt{2}}).$$

We need the following formulas in order to consider the next example.

$$\mathcal{L}^{-1}\{\frac{s}{(s^2+k^2)^2}\} = \frac{1}{2k}tsin(kt) \tag{1}$$

$$\mathcal{L}^{-1}\{\frac{1}{(s^2+k^2)^2}\} = \frac{1}{2k^3}(sin(kt) - ktcos(kt)). \tag{2}$$

Example 2. Find the inverse laplace transform for the function

$$F(s) = \frac{s^2+3}{(s^2+2s+2)^2}.$$

Solution. According to the Rule 2 we are looking for the decomposition of the function $F(s)$ in the form

$$F(s) = \frac{As+B}{(s^2+2s+2)^2} + \frac{Cs+D}{(s^2+2s+2)} = \frac{As+B+Cs^3+2Cs^2+2Cs+Ds^2+2Ds+2D}{(s^2+2s+2)^2}$$

Now we set up the system of linear equations

$$\begin{cases} C = 0 \\ D + 2C = 1 \\ A + 2D + 2C = 0 \\ B + 2D = 3 \end{cases}$$

From the first equation we have $C = 0$ from the second one we have $D = 1$. From the third equation $A = -2$. The last equation imply

$$B = 1.$$

Therefore

$$F(s) = \frac{-2s+1}{((s+1)^2+1)^2} + \frac{1}{((s+1)^2+1)} = -2\frac{s+1}{((s+1)^2+1)^2} + \frac{3}{((s+1)^2+1)^2} + \frac{1}{((s+1)^2+1)}.$$

Using the formulae (2),(3) we obtain

$$\mathcal{L}^{-1}\{F(s)\} = -te^{-t}sin(t) + \frac{3}{2}(sin(t) - tcos(t)) + e^{-t}sin(t).$$