LECTURE 24 (267)

Translation and Partial Fractions.

Let $P(s)$ and $Q(s)$ be the polynomials and degree of the polynomial P is less then degree of the polynomial Q. We consider the rational function

$$R(s) = \frac{P(s)}{Q(s)}. \tag{1}$$

Today we learn some technique for finding the inverse Laplace transform for the function $R(s)$. The following two rules describe the **Partial Fraction decomposition** of $R(s)$. First we need to factorize the polynomial $Q(s)$. Let the polynomial $Q(s)$ has k real roots which we denote as a_k. Each root a_i has the multiplicity n_i. Also let the polynomial $Q(s)$ has 2ℓ complex roots $z_1, z_2, \dots z_\ell, \bar{z}_1, \bar{z}_2, \dots \bar{z}_\ell$. Each root z_i has the multiplicity n_i. then the polynomial $Q(s)$ can be factorized in the form

$$Q(s) = (s-a_1)^{n_1}(s-a_2)^{n_2}\dots(s-a_k)^{n_k}(s-z_1)^{m_1}(s-\bar{z}_1)^{m_1}(s-z_2)^{m_2}(s-\bar{z}_2)^{m_2}\dots(s-z_\ell)^{m_\ell}(s-\bar{z}_\ell)^{m_\ell}$$

The product $(s-z_i)(s-\bar{z}_i)$ can be written into the form

$$(s-z_i)(s-\bar{z}_i) = (s-\tilde{a}_i)^2 + \tilde{b}_i^2.$$

Hence now we can factorize the polynomial $Q(s)$ as

$$Q(s) = (s-a_1)^{n_1}(s-a_2)^{n_2}\dots(s-a_k)^{n_k}((s-\tilde{a}_1)^2+\tilde{b}_1^2)^{m_1}((s-\tilde{a}_2)^2+\tilde{b}_2^2)^{m_2}\dots((s-\tilde{a}_\ell)^2+\tilde{b}_\ell^2)^{m_\ell}.$$

Rule 1. *(Linear Factor Partial Fractions) The part of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $s-a$ of multiplicity n is a sum of n partial fractions, having the form*

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_n}{(s-a)^n}. \tag{2}$$

Rule 2. *(Linear Factor Partial Fractions) The part of the partial fraction decomposition corresponding to the irreducible quadratic factor* $(s-a)^2+b^2$ *of multiplicity* n *is a sum of* n *partial fractions, having the form*

$$\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{((s-a)^2+b^2)^2} + \dots \frac{A_n s + B_n}{((s-a)^2+b^2)^n}.$$

The following theorem is also very useful for finding the inverse Laplace transform:

Theorem 1. *(Translation on the s-Axis) If* $F(s) = \mathcal{L}\{f(t)\}$ *exists for* $s > c$ *then* $\mathcal{L}\{e^{at}f(t)\}$ *exists for* $s > a + c$, *and*

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a). \tag{3}$$

The equivalent formulation of the formula (3) is

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

Formula (3) implies immediately

$$\text{If } f(t) = e^{at}t^n \quad \text{then } F(s) = \frac{n!}{(s-a)^{n+1}},\ s > a, \tag{4}$$

$$\text{If } f(t) = e^{at}cos(kt)\text{then } F(s) = \frac{s-a}{(s-a)^2+k^2},\ s > a, \tag{5}$$

$$\text{If } f(t) = e^{at}sin(kt)\text{then } F(s) = \frac{k}{(s-a)^2+k^2},\ s > a, \tag{6}$$

Example 1.

$$x^{(4)} - x = 0, \quad x(0) = 1, x'(0) = x''(0) = x^{(3)}(0) = 0.$$

Solution. Taking the Laplace transform of the ordinary differential equation we have

$$s^4F(s) - s^3 - F(s) = 0.$$

So

$$F(s) = \frac{s^3}{s^4-1}$$

We need to find the inverse Laplace transform for the function $F(s)$. We can factorize the polynomial $s^4 - 1$ as

$$s^4 - 1 = (s-1)(s+1)(s^2+1)$$

According to the rules one and two we may try to represent our function $F(s)$ as

$$F(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1} = \frac{(A+B)s^3 + (A-B)s^2 + (A+B)s + A - B + Cs^3 + Ds^2 - Cs -}{(s^4-1)}$$

Now we set up the system for unknown coefficients A, B, C, D

$$\begin{cases} A+B+C=1 \\ A-B+D=0 \\ A+B-C=0 \\ A-B-D=0 \end{cases}$$

Solving this system we obtain $A = B = \frac{1}{4}, D = 0, C = \frac{1}{2}$. Hence our system may be represented in the form

$$F(s) = \frac{1}{4}\frac{1}{s-1} + \frac{1}{4}\frac{1}{s+1} + \frac{1}{2}\frac{s}{s^2+1}.$$

Hence the solution to the ordinary differential equation is

$$x(t) = \frac{e^t}{4} + \frac{e^{-t}}{4} + \frac{1}{2}cost.$$

Example 2. Find the inverse Laplace transform for the function

$$F(s) = \frac{s^3}{(s-4)^4}.$$

Solution. According to the rule 1 we are suppose to look for the decomposition of the function $F(s)$ in the form

$$F(s) = \frac{A}{(s-4)^4} + \frac{B}{(s-4)^3} + \frac{C}{(s-4)^2} + \frac{D}{(s-4)} = \frac{A + B(s-4) + C(s-2)^2 + D(s-2)^3}{(s-4)^4}.$$

Now we can set up the system of linear equations

$$\begin{cases} D=1 \\ C-6D=0 \\ B-4C+12D=0 \\ A-4B+4C-8D=0 \end{cases} \begin{cases} D=1 \\ C=6 \\ B=12 \\ A=32 \end{cases}$$

Hence

$$F(s) = \frac{32}{(s-4)^4} + \frac{12}{(s-4)^3} + \frac{6}{(s-4)^2} + \frac{1}{(s-4)}$$

Therefore

$$\mathcal{L}^{-1}\{\frac{s^3}{(s-4)^4}\} = 32t^3e^{4t} + 12t^2e^{4t} + 6te^{4t} + e^{4t}$$

Example 3. Find the inverse Laplace transform for the function

$$F(s) = \frac{3s+5}{s^2-6s+25}$$

Solution. First we transform the function $F(s)$. The polynomial $s^2-6s+25$ has only complex roots so

$$s^2-6s+25 = (s-3)^2+4^2$$

Hence

$$F(s) = \frac{3s+5}{s^2-6s+25} = 3\frac{s-3}{(s-3)^2+4^2} + \frac{7}{2}\frac{4}{(s-3)^2+4^2}.$$

Using the formulas (5) and (6) with $a=3$ and $k=4$ we obtain

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 3e^{3t}cos(4t) + \frac{7}{2}e^{3t}sin(4t).$$

Example 4. Find the inverse Laplace transform for the function

$$F(s) = \frac{1}{s^2+4s+4}.$$

Solution. First we transform the function $F(s)$

$$F(s) = \frac{1}{(s+2)^2}$$

Next we consider the formula (4) with $n=1$ and $a=-2$.We have

$$\mathcal{L}\{e^{-2t}t\} = \frac{1}{(s+2)^2}.$$

So

$$\mathcal{L}^{-1}\{\frac{1}{(s+2)^2}\} = te^{-2t}.$$