LECTURE 23 (267)

Transformation of Initial value Problems.

Example 1. Find the inverse Laplace transform for the following function

$$F(s) = \frac{1}{s(s^2+4)}.$$

Solution. We try to represent the function $F(s)$ as the sum of simple functions

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{(A+B)s^2+Cs+4A}{s(s^2+4)}.$$

So

$$(A+B)s^2+Cs+4A = 1$$

Hence

$$\begin{cases} A+B=0 \\ C=0 \\ 4A=1 \end{cases}$$

The solution to this system of linear equations are $A = \frac{1}{4}, B = -\frac{1}{4}, C = 0$. We have

$$\frac{1}{s(s^2+4)} = \frac{1}{4s} - \frac{1}{4}\frac{s}{s^2+4}.$$

Therefore the inverse transform to the function $F(s)$ is

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+4)}\} = \frac{1}{4} - \frac{1}{4}cos(2t).$$

Example 2. Find the inverse Laplace transform for the following function

$$F(s) = \frac{1}{s(s+1)(s+2)}.$$

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Solution. We try to represent the function $F(s)$ as the sum of simple functions

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)} = \frac{(A+B)s+A}{s^2+s} + \frac{C}{s+2} = \frac{(A+B)s^2+As+2(A+B)s+2A+Cs^2+Cs}{s(s+1)(s+2)}.$$

So $(A+B+C)s^2 + (3A+2B+C)s + 2A = 1$ and we obtain the system of linear equations

$$\begin{cases} A+B+C=0 \\ 3A+2B+C=0 \\ 2A=1 \end{cases} \begin{cases} B+C=-\frac{1}{2} \\ 2B+C=-\frac{3}{2} \\ A=\frac{1}{2} \end{cases} \begin{cases} C=\frac{1}{2} \\ B=-1 \\ A=\frac{1}{2} \end{cases}$$

Therefore we have

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+2)}.$$

The inverse transform to the function $F(s)$ is

$$\mathcal{L}^{-1}\{\frac{1}{s(s+1)(s+2)}\} = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}.$$

Example 3. Solve the initial value problem by the method of the Laplace transform:

$$x'' + 4x' + 3x = 1, \quad x(0) = x'(0) = 0.$$

Solution. Applying the Laplace transform to the both sides of the ordinary differential equation we obtain

$$s^2F(s) + 4sF(s) + 3F(s) = \frac{1}{s}.$$

So

$$F(s) = \frac{1}{s(s^2+4s+3)} = \frac{1}{s(s+1)(s+3)}$$

We try to represent the function $\frac{1}{s(s+1)(s+3)}$ as the sum of simple functions

$$\frac{1}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} = \frac{(A+B)s+A}{s^2+s} + \frac{C}{s+3}$$

$$= \frac{(A+B)s^2+As+3(A+B)s+3A+Cs^2+Cs}{s(s+1)(s+3)}.$$

Hence we have $(A+B+C)s^2 + (4A+3B+C)s + 3A = 1$. So

$$\begin{cases} A+B+C=0 \\ 4A+3B+C=0 \\ 3A=1 \end{cases} \begin{cases} B+C=-\frac{1}{3} \\ 3B+C=-\frac{4}{3} \\ A=\frac{1}{3} \end{cases} \begin{cases} C=-\frac{1}{6} \\ B=-\frac{1}{2} \\ A=\frac{1}{3} \end{cases}$$

Now we have the following

$$F(s) = \frac{1}{s(s^2+4s+3)} = \frac{1}{3s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}$$

So

$$x(t) = \frac{1}{3} - \frac{e^{-t}}{2} + \frac{e^{-3t}}{6}.$$