

CONVERGENCE TESTS

By the alternating series we mean the series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n.$$

where $b_n \geq 0$ for all n .

The Alternating Series Test. *If the alternating series satisfies*

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad b_n > 0$$

satisfies

$$b_{n+1} \leq b_n \quad \text{for all } n$$
$$\lim_{n \rightarrow \infty} b_n = 0$$

then the series converges.

Alternating series estimation theorem. *If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies*

$$b_{n+1} \leq b_n \quad \text{for all } n$$
$$\lim_{n \rightarrow \infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example 1. Test the infinite series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ for convergence or divergence.

Solution. We use the Alternating Series Test. Here $b_n = \frac{n}{n^2+1} > 0$. Obviously $\lim_{n \rightarrow \infty} b_n = 0$ and finally

$$b_{n+1} - b_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{-n^2 - n - 1}{((n+1)^2+1)(n^2+1)} < 0.$$

. So all conditions of the theorem holds true and our infinite series in convergent.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-T}\mathcal{E}\mathcal{X}$

Definition. A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Example 2. Determine whether the series $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n^2}$ is absolutely convergent.

Solution. We use the comparison test. Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This infinite series is absolutely convergent since this is p -series with $p = 2 > 1$. Note that

$$|\sin(2n)| \leq 1$$

So

$$\left| \frac{\sin(2n)}{n^2} \right| \leq \frac{1}{n^2}.$$

Therefore by comparison test our infinite series is convergent.

Theorem. If the series absolutely convergent, then it convergent

It is not difficult at all to find an example of the infinite series which is convergent but not the absolutely convergent. Really we consider the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$. This infinite series is convergent. Really let us check the conditions of the Alternating Series Test. We have $b_n = \frac{1}{n}$. Obviously $b_n > 0$, $\frac{1}{n+1} < \frac{1}{n}$. And of course $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$. So all conditions of the test holds true. Therefore the infinite series is convergent. On the other hand the infinite series

$$\sum_{n=1}^{\infty} |(-1)^{n-1} \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent since this is the p -series with $p = 1$.

Ratio Test. A.) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

B.) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 3. Determine whether the series $\sum_{n=1}^{\infty} \frac{(n+2)!}{n!10^n}$ is absolutely convergent.

Solution. We apply the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+3)!}{(n+1)!10^{n+1}}}{\frac{(n+2)!}{n!10^n}} = \frac{(n+3)!n!10^n}{(n+2)!(n+1)!10^{n+1}} = \frac{n+3}{10(n+1)}.$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{10}$$

and the infinite series is convergent.