LECTURE 22 (267)

Laplace transform.

Theorem 1.. *Suppose that the function $f(t)$ is continuous and piecewise smooth for $t \geq 0$ and of exponential order as $t \to \infty$, so that there exist nonnegative constants M, C, T such that*

$$|f(t)| \leq Me^{ct} \quad \text{for } t \geq T. \tag{1}$$

Then $\mathcal{L}\{f'(t)\}$ exists for $s > c$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f'(t)\} - f(0) = sF(s) - f(0). \tag{2}$$

for the Laplace transform of $f''(t)$ we have

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0). \tag{3}$$

Corollary. *(Transforms of Higher Derivatives) Suppose that the functions $f, f', f'', \ldots, f^{(n-1)}$ are continuous and piecewise smooth for $t > 0$ and that each of these functions satisfies the condition (1) with the same values of M and c. Then $\mathcal{L}\{f^{(n)}\}$ exists when $s > c$ and*

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)} - f^{(n-1)}(0).$$

Example 1. Use the Laplace transform technique to solve the initial value Problem

$$x'' + 4x = 0, \quad x(0) = 5,\ x'(0) = 0.$$

*Solution.*Applying the Laplace transform to the both sides of the equation we obtain

$$s^2F(s) + 4F(s) - 5s = 0$$

So

$$F(s) = \frac{5s}{s^2+4}.$$

Then

$$x(t) = \mathcal{L}^{-1}\{F(s)\} = 5cos(2t).$$

Example 2. Use the Laplace transform technique to solve the initial value Problem

$$x'' - x' - 2x = 0, \quad x(0) = 0,\, x'(0) = 2.$$

*Solution.*Applying the Laplace transform to the both sides of the equation we obtain

$$s^2F(s) - sF(s) - 2F(s) - 2 = 0$$

Then

$$F(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s+1)(s-2)} = -\frac{2}{3}(\frac{1}{s+1} - \frac{1}{s-2})$$

Hence

$$x(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}.$$

Example 3. Use the Laplace transform technique to solve the initial value Problem

$$x'' + x = sin(2t), \quad x(0) = 0,\, x'(0) = 0.$$

*Solution.*Applying the Laplace transform to the both sides of the equation we obtain

$$s^2F(s) + F(s) = \frac{1}{s^2+4}.$$

Then

$$F(s) = \frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}(\frac{1}{s^2+1} - \frac{1}{s^2+4})$$

Hence

$$x(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}sin(t) - \frac{1}{6}sin(2t).$$

Example 4. Use the Laplace transform technique to solve the initial value Problem

$$x'' + 3x' + 2x = t, \quad x(0) = 0,\, x'(0) = 2.$$

*Solution.*Applying the Laplace transform to the both sides of the equation we obtain

$$s^2F(s) + 3sF(s) + 2F(s) - 2 = \frac{1}{s^2}.$$

Then

$$F(s) = \frac{2}{(s^2+3s+2)} + \frac{1}{s^2(s^2+3s+2)} \tag{5}$$

We find the inverse Laplace transform for each term in the right hand side of (5) separately

$$\frac{2}{(s^2+3s+2)} = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{As+2A+Bs+B}{(s+2)(s+1)}.$$

Then

$$A+B=0, \quad B+2A=2.$$

So $A=2$ and $B=-2$. So

$$\frac{2}{(s^2+3s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$

Hence

$$\mathcal{L}^{-1}\{\frac{2}{(s^2+3s+2)}\} = 2e^{-t} - 2e^{-2t}.$$

For the second term

$$\frac{1}{s^2(s^2+3s+2)} = \frac{D}{s^2} + \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} =$$

$$\frac{D+As}{s^2} + \frac{(B+C)s+(2C+B)}{s^2+3s+2} = \frac{D(s^2+3s+2)+A(s^3+3s^2+2s)+(B+C)s^3+(2C+B)s^2}{s^2(s^2+3s+2)}. \tag{6}$$

We try to find the coefficients A, B, C, D. From (7) we obtain the system of O.D.E

$$\begin{cases} 2D = 1 \\ 3A + D + 2C + B = 0 \\ A + B + C = 0 \\ 3D + 2A = 0 \end{cases}$$

So $D = \frac{1}{2}, A = -\frac{1}{6} C = \frac{1}{6}, B = -\frac{1}{3}$. Hence

$$\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+3s+2)}\} = \frac{t}{2} - \frac{1}{6} - \frac{1}{3}e^{-2t} + \frac{1}{6}e^{-t}$$

The solution is

$$x(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{t}{2} - \frac{1}{6} - \frac{1}{3}e^{-2t} + \frac{1}{6}e^{-t} + 2e^{-t} - 2e^{-2t}.$$