LECTURE 2‘ ‘ ‘ (267)

We consider the first order differential equation

$$\frac{dy}{dt} = f(t) \tag{1}$$

The general solution to this equation is given by the formula

$$y(t) = \int f(t)dt + C = \int_{t_0}^{t} f(s)ds + C \tag{2}$$

If we consider the initial value problem

$$\frac{dy}{dt} = f(t), \quad y(t_0) = y_0$$

The solution to this initial value problem is

$$y(t) = \int f(t)dt + C = \int_{t_0}^{t} f(s)ds + y_0 \tag{3}$$

We consider several examples

Example 1. Solve the initial value problem

$$\frac{dy}{dt} = 2t + 1 \quad y(0) = 1$$

Solution. Using the formula (3) we have

$$y(t) = \int_{0}^{t} (2s+1)ds + 1 = t^2 + t + 1.$$

Example 2. Solve the initial value problem

$$\frac{dy}{dt} = cos(2t) \quad y(0) = 0.$$

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Solution. Using the formula (3) we have

$$y(t) = \int_0^t cos(2s)ds = -\frac{sin(2t)}{2}.$$

Example 3. Solve the initial value problem

$$\frac{dy}{dt} = t\sqrt{t^2+9} \quad y(-4) = 0.$$

Solution. Using the formula (3) we have

$$y(t) = \int_{-4}^t s\sqrt{s^2+9}ds = \frac{1}{3}(s^2+9)^{\frac{3}{2}}|_{-4}^t = \frac{1}{3}((t^2+9)^{\frac{3}{2}} - 5^3).$$

Velocity and Acceleration

We consider an object moving along the strait line (x-axis) and denote the position of an object ta time moment t as $x(t)$. The **velocity** of an object v is given by formula

$$v(t) = \frac{dx}{dt}.$$

and the acceleration $a(t)$ is

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

By the Newton's law of motion we have

$$ma = F(t) \tag{4}$$

Her m is a mass of an object, F is an external force which is acting on object. We can rewrite (4) as

$$mx'' = F \tag{5}$$

Example 4. Find a position of a moving particle with given acceleration $a(t)$, initial position x_0 and initial velocity v_0. Here $a(t) = 50, v_0 = 10, x_0 = 20$.

Solution.

$$x''(t) = 50, \quad x(0) = 20, \quad x'(0) = 10..$$

We have $x(t) = 25t^2 + C_1 t + C_2$. Since $x(0) = 20$ we have $x(0) = 20 = C_2$. So $x(t) = 25t^2 + C_1 t + 20$. Next $x'(t) = 50t + C_1$. Since $x'(0) = 10$ we have $x'(0) = 10 = C_1$. So the formula for a motion is

$$x(t) = 25t^2 + 10t + 20.$$

Example 5. Find a position of a moving particle with given acceleration $a(t)$, initial position x_0 and initial velocity v_0. Here $a(t) = 50sin(50t), v_0 = -10, x_0 = 8$.

Solution.

$$x''(t) = 50sin(50t), \quad x(0) = 8, \quad x'(0) = -10.$$

We have

$$x'(t) = \int_0^t 50sin(50s)ds - 10 = -cos(50s)|_0^t - 10 = -cos(50t) - 9$$

Next

$$x(t) = \int_0^t (-cos(50s) - 9)ds + 8 = (-\frac{1}{50}sin(50s) - 9s)|_0^t + 8 = (-\frac{1}{50}sin(50t) - 9t) + 8.$$

Vertical motion in gravitational vector field.

The gravitational force given by formula

$$F = mg$$

Here m is the mass of an object and g is the gravitational constant. If we measure the distance in meters time in seconds then $g = 9.8$ m$/s^2$. If we measure distance in foots then $g = 32$ ft$/s^2$.

If our coordinate axis directed upward the equation of a motion is

$$x'' = -g$$

So

$$x(t) = -\frac{1}{2}gt^2 + v_0 + x_0.$$

where x_0 is a position of object at time moment $t = 0$ and v_0 is the velocity at this moment.

Example a ball dropped from the top of the building $400ft$ height. How long does it take for a ball to reach the ground? With what speed it strike the ground.

Solution. Denote the time when a ball strike the earth as T and the speed as V. Note that $v_0 = 0$ and $x_0 = 400$, $g = 32$. So the equation of the motion is

$$x(t) = -16t^2 + 400.$$

Note that $x(T) = 0$ so

$$x(T) = 0 = -16T^2 + 400.$$

Solving this equation we have $T = 5$. Next the equation for velocity is $x'(t) = -32t$. So $V = |x'(T)| = 160$.