CONVERGENCE TESTS.

Integral Test. *Suppose f is continuous, positive decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words*
a *If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*
b *If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.*

Theorem. *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.*

The comparison Test. . *Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms*
a *If $\sum_{n=1}^{\infty} b_n$ is convergent, and $a_n \leq b_n$ for all n starting from some N then $\sum_{n=1}^{\infty} a_n$ is also convergent.*
b *If $\sum_{n=1}^{\infty} b_n$ is divergent, and $a_n \geq b_n$ for all n starting from some N then $\sum_{n=1}^{\infty} a_n$ is also divergent.*

The limit comparison test. *Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with the positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$ then either both series converge or both series diverge.

Remainder estimate for the integral test. *If $\sum_{n=1} a_n$ converges by the Integral Test and $R_n = s - s_n$, then*

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

Example 1. Determine whether the series $\sum_{n=1}^{\infty}(\frac{2}{n\sqrt{n}} + \frac{3}{n^3})$ is convergent or divergent.

Solution. We can use the limit comparison test . We consider the infinite series $\sum \frac{1}{n^{\frac{3}{2}}}$. This is $p-$ series with $p = \frac{3}{2}$.

Obviously

$$\lim_{n\to+\infty} \frac{(\frac{2}{n\sqrt{n}} + \frac{3}{n^3})}{1n^{\frac{3}{2}}} = 2.$$

So by the limit comparison test the infinite series in convergent.

Example 2. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ is convergent or divergent.

Solution. We use the comparison test. It is know that the infinite series $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ is divergent. Obviously $\frac{1}{2\sqrt{n}} \geq \frac{1}{\sqrt{n}-1}$ for all sufficiently large n. Therefore the infinite series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ is divergent.

Example 3. Determine whether the series $\sum_{n=2}^{\infty} sin(\frac{1}{n})$ is convergent or divergent.

Solution. We use the limit comparison test. The infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Since the limit

$$\lim_{n\to\infty} \frac{sin(\frac{1}{n})}{\frac{1}{n}} = 1$$

the infinite series $\sum_{n=2}^{\infty} sin(\frac{1}{n})$ is divergent.

Example 4. Find all values of p for which the infinite series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent.

*Solution.*Obviously for $p \leq 0$ the infinite series in divergent. Now we use the integral test

$$\int_2^N \frac{1}{x(\ln x)^p} dx = \frac{\ln x}{(\ln x)^p}|_2^N + p\int_2^N \frac{\ln x(\ln x)^{-p-1}}{x} dx = \frac{\ln x}{(\ln x)^p}|_2^N + p\int_2^N \frac{1}{x(\ln x)^p} dx.$$

Hence

$$(1-p)\int_2^N \frac{1}{x(\ln x)^p} dx = \frac{\ln x}{(\ln x)^p}|_2^N.$$

So

$$\int_2^N \frac{1}{x(\ln x)^p} dx = \frac{1}{1-p}\frac{\ln x}{(\ln x)^p}|_2^N.$$

Now

$$\text{for } p > 1 \quad \lim_{N\to+\infty} \int_2^N \frac{1}{x(\ln x)^p} dx = \frac{1}{p-1}\frac{\ln 2}{(\ln 2)^p}.$$

So for $p > 1$ the infinite series is convergent.

$$\text{for } p < 1 \quad \lim_{N \to +\infty} \int_2^N \frac{1}{x(\ln x)^p} dx = \infty.$$

So for $p < 1$ the infinite series is divergent.