LECTURE 20 (267)

Laplace transform.

Definition. *Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined as follows:*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

for all values of s for which the improper integral converges.

We remind that by **improper integral** over an infinite interval we mean

$$\int_0^\infty g(t)dt = \lim_{b\to\infty} \int_0^b g(t)dt.$$

Now we state some properties of the Laplace transform.

Theorem 1. *(Linearity of the Laplace transform) If a and b are the constants, then*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

for all s such that the Laplace transforms of the function f and g both exist.

Definition. *The function f is said to be of exponential order as $t \to +\infty$ if there exist nonnegative constants M, c and T such that*

$$|f(t)| \leq Me^{ct} \quad \mathit{for} t \geq T. \tag{1}$$

Definition. *The function $f(t)$ is said to be piecewise continuous on the bounded interval $a \leq t \leq b$ provided that $[a, b]$ can be subdivided into finitely many abutting subintervals in such way that:*

1. *f continuous in the interior of each of these subintervals: and*

2. *$f(t)$ has a finite limit as t approachers each endpoint of each subinterval from the interior.*

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Theorem 2. *(Existence of the Laplace transform) If the function f is piecewise continuous for $t \geq 0$ and is of exponential order as $t \to +\infty$ then its Laplace transform $F(s)$ exists. More precisely, if f is piecewise continuous and satisfies the condition (1), then $F(s)$ exists for all $s > c$..*

Corollary. *If $f(t)$ satisfies the hypotheses of Theorem 2, then*

$$\lim_{s \to \infty} F(s) = 0$$

Theorem 3. *(Uniqueness of Inverse Laplace transform) Suppose that the functions $f(t)$ and $g(t)$ satisfy the hypothesis of Theorem 2, so that there Laplace transforms $f(s)$ and $G(s)$ both exist. If $F(s) = G(s)$ for all $S \geq c$ then $f(t) = g(t)$ wherever on $[0, +\infty)$ both f and g are continuous.*

Table of Laplace Transforms.

$f(t)$	$F(s)$	*Domain*
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^a (a > -1)$	$\frac{\Gamma(a+1)}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > 0$
$cos(kt)$	$\frac{s}{s^2+k^2}$	$s > 0$
$cosh(kt)$	$\frac{s}{s^2-k^2}$	$s > \|k\|$
$sinh(kt)$	$\frac{k}{s^2-k^2}$	$s > \|k\|$
$u(t-a)$	$\frac{e^{-as}}{s}$	$s > 0$

Definition.. *Gamma function $\Gamma(x)$ defined for $x > 0$ by the formula*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

It is known that

$$\Gamma(1) = 1$$

and that

$$\Gamma(x+1) = x\Gamma(x).$$

Example 1. Find the Laplace transform for the function $f(t) = t$.

Solution.

$$F(s) = \lim_{b\to+\infty}\int_0^b e^{-st}tdt = \lim_{b\to+\infty}(-\frac{e^{-st}}{s}t|_0^b + \frac{1}{s}\int_0^b e^{-st}dt) = \lim_{b\to+\infty}(-\frac{e^{-sb}}{s} - \frac{e^{-st}}{s^2}|_0^b) \tag{2}$$

On the other hand, one can easily see that for $s < 2$ there is no limit in (2). Assuming that $s > 0$ we have

$$F(s) = \frac{1}{s^2}.$$

Example 2. Find the Laplace transform for the function $f(t) = cost$.
Solution.

$$F(s) = \lim_{b\to+\infty}\int_0^b e^{-st}cos(t)dt$$

Now we find the antiderivative for the function $e^{-st}cos(t)$:

$$\int e^{-st}cos(t)dt = -\frac{1}{s}cos(t)e^{-st} - \frac{1}{s}\int e^{-st}sin(t)dt =$$
$$-\frac{1}{s}cos(t)e^{-st} + \frac{1}{s^2}e^{-st}sin(t) - \frac{1}{s^2}\int e^{-st}cos(t)dt.$$

So

$$\frac{s^2+1}{s^2}\int e^{-st}cos(t)dt = -\frac{1}{s}cos(t)e^{-st} - \frac{1}{s^2}e^{-st}sin(t)$$

And

$$\int e^{-st}cos(t)dt = \frac{s^2}{s^2+1}(-\frac{1}{s}cos(t)e^{-st} - \frac{1}{s^2}e^{-st}sin(t))$$

Using this formula we obtain that for $s > 0$

$$F(s) = \lim_{b\to+\infty}\int_0^b e^{-st}cos(t)dt = \lim_{b\to+\infty}\frac{s^2}{s^2+1}(-\frac{1}{s}cos(t)e^{-st} - \frac{1}{s^2}e^{-st}sin(t))|_0^b = \frac{s}{s^2+1}.$$

Example 3. Find the Laplace transform for the function $f(t) = sinh(t) = \frac{1}{2}(e^t - e^{-t})$.
Solution. By the definition of the Laplace transform we have

$$F(s) = \lim_{b\to+\infty}\int_0^b sinh(t)e^{-st}dt = \lim_{b\to+\infty}\int_0^b \frac{1}{2}(e^t - e^{-t})e^{-st}dt = \lim_{b\to+\infty}\int_0^b \frac{1}{2}(e^{(1-s)t} - e^{-(1+s)t})dt =$$
$$\lim_{b\to+\infty}\frac{1}{2}(\frac{e^{(1-s)t}}{1-s} - \frac{e^{-(1+s)t}}{-(1+s)})|_0^b = \lim_{b\to+\infty}\frac{1}{2}(\frac{e^{(1-s)b}}{1-s} - \frac{e^{-(1+s)b}}{-(1+s)} - (\frac{1}{1-s} - \frac{1}{-(1+s)})) = \frac{1}{s^2-1}.$$

Here in order to pass to the limit we assumed $s > 1$.

Example 4. Find the Laplace transform for the function

$$f(t) = \begin{cases} 1 & \text{for } t \in [1,2] \\ 0 & \text{for } t \notin [1,2] \end{cases}$$

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Solution. By the definition of the Laplace transform

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_1^2 e^{-st}dt = \frac{e^{-st}}{-s}|_1^2 = \frac{e^{-2s}}{s} - \frac{e^{-s}}{s}.$$