PARAMETRIC SURFACES, SURFACE INTEGRALS

We learned that a surface S mathematically can be represented
1. As a graph of a function : $z = f(x,y)$ where $(x,y) \in D$.
2. Or it might be introduced using the equation $F(x,y,z) = 0$.
The third way mathematically describe the surface is the following:
Let v and u be the independent variables. We consider the function

$$r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}, \tag{1}$$

where (u,v) are from a region D on the plane.

In that case a point with the coordinates (x_0, y_0, z_0) is on the surface S if and only if there exists $(u_0, v_0) \in D$ such that

$$x_0 = x(u_0,v_0), y_0 = y(u_0,v_0), z_0 = z(u_0,v_0). \tag{2}$$

Let us consider the sphere $x^2 + y^2 + z^2 = R^2$. Then the parametric representation for a sphere is

$$r(\phi,\theta) = Rsin\phi cos\theta\vec{i} + Rsin\phi sin\theta\vec{j} + Rcos\phi\vec{k},$$

where $(\phi,\theta) \in [0,\pi] \times [0,2\pi)$.

The parametric representation of the cylinder $x^2 + z^2 = R^2$ is

$$r(\theta,u) = Rcos\theta\vec{i} + u\vec{j} + Rsin\theta\vec{k},$$

where $(\theta,u) \in [0,2\pi) \times \mathbb{R}^1$.

If the surface is given as a graph of function $z = f(x,y)$ we can easily convert it in the parametric form

$$r(u,v) = u\vec{i} + v\vec{j} + f(u,v)\vec{k}, \quad (u,v) \in D.$$

Denote

$$r_u(u,v) = \frac{\partial x}{\partial u}(u,v)\vec{i} + \frac{\partial y}{\partial u}(u,v)\vec{j} + \frac{\partial z}{\partial u}(u,v)\vec{k},$$

$$r_v(u,v) = \frac{\partial x}{\partial v}(u,v)\vec{i} + \frac{\partial y}{\partial v}(u,v)\vec{j} + \frac{\partial z}{\partial v}(u,v)\vec{k}.$$

The Normal Vector to the parametric surface at point (x_0, y_0, z_0) is given by formula

$$\vec{N} = r_u(u_0, v_0) \times r_v(u_0, v_0).$$

Example 1. Find equation of the tangent plane to a parametric surface $x = u + v, y = 3u^2 z = u - v$ at the point $(2, 3, 0)$.

Solution. First we need to find u_0, v_0 such that

$$\begin{cases} 2 = u_0 + v_0 \\ 3 = 3u_0^2 \\ 0 = u_0 - v_0 \end{cases}$$

Adding to the first equation the second one we have $2 = 2u_0$. Hence $u_0 = 1$ and from the third equation we have $v_0 = 1$. Obviously the second equation holds true also. Next we need the vectors r_u and r_v. $r_u(u,v) = \vec{i} + 6u\vec{j} + \vec{k}$, $r_v(u,v) = \vec{i} - \vec{k}$. Hence

$$r_u(1,1) = \vec{i} + 6\vec{j} + \vec{k},$$

$$r_v(1,1) = \vec{i} - \vec{k}.$$

Now the normal vector

$$\vec{N} = r_u(1,1) \times r_v(1,1) = (-6, 2, -6).$$

Then the equation for a tangent plane is

$$-6(x-2) + 2(y-3) - 6z = 0.$$

Definition. *If a smooth surface S is given by the equation*

$$r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k} \quad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D then the surface area of S is

$$A(S) = \int\int_D |r_u \times r_v| dA.$$

Example 2. Find the area of the surface $x = uv, y = u + v, z = u - v$ and $d = \{(u, v)|u^2 + v^2 \leq 1\}$.

Solution. We have $r_u = v\vec{i} + \vec{j} + \vec{k}, \quad r_v = u\vec{i} + \vec{j} - \vec{k}$. Then

$$r_u \times r_v = -2\vec{i} + (u + v)\vec{j} + (v - u)\vec{k}.$$

and $|r_u \times r_v| = \sqrt{4 + 2(u^2 + v^2)}$. Therefore

$$A(S) = \int\int_D \sqrt{4 + 2(u^2 + v^2)} dA$$

In order to evaluate this integral we use the polar coordinate system

$$A(S) = \int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2} r dr d\theta = \frac{1}{6} \int_0^{2\pi} (4 + 2r^2)^{\frac{3}{2}} |_0^1 d\theta = \frac{\pi}{3}(6^{\frac{3}{2}} - 8).$$

Next we consider more general situation

Definition. *If a smooth surface S is given by the equation*

$$r(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k} \quad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D. Let $f(x, y, z)$ be a function with domain on the surface S. Then the surface integral $\int_S f(x, y, z) dS$ is

$$\int\int_S f(x, y, z) dS = \int\int_D f(x(u, v), y(u, v), z(u, v)) |r_u \times r_v| dA.$$

Example 3. Evaluate the surface integral $\int\int_S \sqrt{1 + x^2 + y^2} dS$ where $r(u, v) = ucosv\vec{i} + usinv\vec{j} + v\vec{k}$, $0 \leq u \leq 1, 0 \leq v \leq \pi$.

Solution. We have

$$r_u = cosv\vec{i} + sinv\vec{j}, \quad r_v = -usinv\vec{i} + ucosv\vec{j} + \vec{k}.$$

and

$$r_u \times r_v = sinv\vec{i} - cosv\vec{j} + u\vec{k}.$$

Hence

$$|r_u \times r_v| = \sqrt{1 + u^2}.$$

Then we have

$$\int\int_S \sqrt{1+x^2+y^2}dS = \int\int_D \sqrt{1+u^2}\sqrt{1+u^2}dA = \int_0^\pi\int_0^1 (1+u^2)dudv =$$
$$\int_0^\pi (u+\frac{u^3}{3})|_0^1 dv = \int_0^\pi \frac{4}{3}dv = \frac{8\pi}{3}.$$

Let S be a smooth surface, denote by $\vec{n}=\vec{n}(x,y,z)$ a unit normal vector to the surface S at the point (x,y,z). By the unit vector we mean the vector of length one. Suppose that the vector function $\vec{n}$ is continuous Then there are only two possible choices for the function $\vec{n}$

$$\vec{n}(x,y,z) \quad or \quad -\vec{n}(x,y,z).$$

If the choice of the vector field is done, the surface S is called oriented. Let S be the oriented surface, and $\mathbf{F}$ be a vector field on S. Then Flux of $\mathbf{F}$ across S is

$$\int\int_S F\cdot dS = \int\int_S \mathbf{F}\cdot ndS.$$