GREEN'S THEOREM. *div* AND *curl* OPERATORS

Green's Theorem. *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If the functions P and Q have a continuous partial derivatives on an open region that containes D then*

$$\int_C Pdx + Qdy = \int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{1}$$

Definition 1. *A region which boundary consists of one piecewise-smooth, simple closed curve is called* **simple region.**

Example. The disk $D = \{(x, y)|x^2 + y^2 \le R^2\}$ is the simplke region. The boundary of the disk consists of the single curve - the circle $x^2+y^2 = R^2$. On the other hand the region $D_1 = \{(x, y)|\frac{R^2}{4} \le x^2+y^2 \le R^2\}$ is not the simple region, since the boundary of this region consists of two circles $x^2 + y^2 = \frac{R^2}{4}$ and $x^2 + y^2 \le R^2$.

Theorem. *Let D be a simple region bounded by the curve C then the area of D given by formula*

$$Area = \oint xdy = -\oint ydx = \frac{1}{2}\oint xdy - ydx. \tag{2}$$

Example 1. Use the Green's theorem in order to evaluate the integral $\int_C x^2ydy - 3y^2dy$ where C is the circle $x^2 + y^2 = 1$.

*Solution.*In our case $P = x^2y$ and $Q = -3y^2$. Then $\frac{\partial P}{\partial y} = x^2$ and $\frac{\partial Q}{\partial x} = 0$. Let $D = \{(x, y)|x^2 + y^2 \le 1\}$. By Green's theorem we have

$$\int_C x^2ydy - 3y^2dy = \int\int_D -x^2dA.$$

In order to evaluate this integral we use the polar coordinate system

$$\int\int_D -x^2dA = -\int_0^{2\pi}\int_0^1 r^3cos^2\theta drd\theta = -\int_0^{2\pi}\frac{r^4}{4}cos^2\theta|_0^1d\theta =$$

$$-\frac{1}{4}\int_0^{2\pi} cos^2\theta d\theta = -\frac{1}{4}(\frac{1}{2}\theta + \frac{1}{4}sin(2\theta))|_0^{2\pi} = -\frac{\pi}{4}.$$

Example 2. Find the area of region bounded by curve with vector equation $r(t) = cost\vec{i} + sin^3t\vec{j}$ where $0 \le t \le 2\pi$.

Solution. In order to evaluate the area of a region bounded by this curve we use formula (2). In our case $x(t) = cost$ and $y(t) = sin^3t$. Thus

$$Area = \oint xdy = \int_0^{2\pi} cos(t)3sin^2(t)cos(t)dt = 3\int_0^{2\pi}(cos(t)sin(t))^2dt = \frac{3}{4}\int_0^{2\pi} sin^2(2t)dt$$
$$= \frac{3}{8}(t - \frac{1}{4}sin(4t))|_0^{2\pi} = \frac{3\pi}{4}.$$

Let $\mathbf{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field over $\mathbb{R}^3$. Then

$$curl\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}. \quad (3)$$

and

$$div\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \quad (4)$$

In order to memorize formula (3) it is convinient to use the cross product

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

It is known that

$$curl(\nabla f) = 0$$

and

$$div\, curl\mathbf{F} = 0.$$

Also we have the following important theorem

Theorem. *If* $\mathbf{F}$ *is a smooth vector field over* $\mathbb{R}^3$ *such that* $curl\mathbf{F} = 0$ *then vector field* $\mathbf{F}$ *is conservative.*

Example 3. For the vector field $\mathbf{F} = x\vec{i} + y\vec{j} + (x+y+z)\vec{k}$ find $curl\mathbf{F}$ and $div\mathbf{F}$.

Solution. We have $P(x,y,z) = x, Q(x,y,z) = y, R(x,y,z) = x+y+z$. Then

$$P_x = 1, \quad Q_y = 1, \quad R_z = 1.$$

So

$$div\mathbf{F} = 1 + 1 + 1 = 3.$$

On the other hand

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 1 - 0 = 1;$$

$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0 - 1 = -1;$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 0 = 0;$$

Hence

$$curl\mathbf{F} = \vec{i} - \vec{j}.$$