

LECTURE 15 (267)

Undetermined coefficients.

We consider the linear ordinary differential equation with constant coefficients

$$a_0y^{(n)} + a_1y^{(n-1)} \dots + a_{n-1}y' + a_ny = f. \quad (1)$$

Suppose that the right hand side of this equation has the form

$$f = P_m(t)e^{\gamma t}$$

We are looking for a particular solution to equation (1). Let s be zero if γ is not the root of characteristic equation. If γ is the root to characteristic equation the s equal to the multiplicity of this root. Then there exists the particular solution to equation (1) in the form

$$y_p = t^s Q_m(t)e^{\gamma t}$$

Here $Q_m(t)$ is a polynomial of order m with unknown coefficients

$$Q_m(t) = b_0 + b_1t + \dots + b_{m-1}t^{m-1} + b_mt^m.$$

If the right hand side has the form

$$f(t) = e^{\alpha t}(P_m(t)\cos\beta t + R_m(t)\sin(\beta t))$$

where

$$P_m(t) = p_0 + p_1t + \dots + p_{m-1}t^{m-1} + p_mt^m, \quad R_m(t) = r_0 + r_1t + \dots + r_{m-1}t^{m-1} + r_mt^m.$$

We set $\gamma = \alpha + i\beta$. Let s be zero if γ is not the root of characteristic equation. If γ is the root to characteristic equation the s equal to the multiplicity of this root. Then there exists the particular solution to equation (1) in the form

$$y_p = t^s e^{\alpha t}(Q_m(t)\cos(\beta t) + G_m(t)\sin(\beta t))$$

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where

$$Q_m(t) = q_0 + q_1 t + \dots + q_{m-1} t^{m-1} + q_m t^m, \quad G_m = G_0 + G_1 t + \dots + G_{m-1} t^{m-1} + G_m t^m.$$

Example 1.

Find a general solution to equation

$$y'' + 16y = e^{3t}$$

Solution. The characteristic polynomial for this problem is

$$r^2 + 16$$

it has two roots $r_{1,2} = \pm 4i$. The linearly independent solutions which correspond to these roots are

$$y_1(t) = \cos(4t), \quad y_2(t) = \sin(4t).$$

We look for the particular solution in the form

$$y_p(t) = t^s Q_0(t) e^{3t}$$

The $\gamma = 3$ is not the root to the characteristic equation, so $s = 0$. $Q_0(t)$ the polynomial of the order zero. Hence we should look for our particular solution in the form

$$y_p(t) = q_0 e^{3t}$$

We observe

$$y_p''(t) = 9q_0 e^{3t}.$$

and

$$y_p'' + 16y_p = 9q_0 e^{3t} + 16q_0 e^{3t} = e^{3t}$$

We have

$$q_0 = \frac{1}{25}.$$

The general solution is

$$y(t) = C_1 \cos(4t) + C_2 \sin(4t) + \frac{1}{25} e^{3t}.$$

Example 2.

Find a general solution to equation

$$y^{(3)} + 4y' = 3t - 1$$

Solution. The characteristic polynomial is

$$r^3 + 4r = r(r - 2i)(r + 2i)$$

Hence the general solution to the homogeneous equation is $Y(t) = c_1 + c_2 \cos(2t) + c_3 \sin(2t)$. Next we find the particular solution

$$y_p = t^s Q_1(t) e^{0t}$$

In this situation $\gamma = 0$ is the root to the characteristic equation of the multiplicity one. So $s = 1$. and $Q_1 = q_0 + tq_1$. Finally we are looking for particular solution in the form

$$y_p(t) = (tq_0 + t^2q_1)$$

We observe that

$$y_p'(t) = q_0 + 2q_1t, \quad y_p'''(t) = 0$$

Hence

$$4(q_0 + 2q_1t) = 3t - 1.$$

and $q_0 = -\frac{1}{4}$, $q_1 = \frac{3}{8}$. The general solution is

$$y(t) = c_1 + c_2 \cos(2t) + c_3 \sin(2t) \frac{t}{4} + \frac{3t^2}{8}.$$

Example 3.

Find a general solution to equation

$$y^{(3)} - y = e^t + 7$$

Solution. The characteristic equation is

$$r^3 - 1 = (r - 1)(r^2 + r + 1) = (r - 1)\left(r - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)\left(r - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)$$

So the general solution to the homogeneous equation is

$$Y(t) = c_1 e^t + c_2 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right).$$

We split the particular solution into the sum of two functions

$$y_p(t) = y_{p,1}(t) + y_{p,2}(t)$$

where

$$y_{p,1}^{(3)} - y_{p,1} = e^t$$

and

$$y_{p,2}^{(3)} - y_{p,2} = 7$$

Obviously one can take $y_{p,2} = -7$. And we are looking for the function

$$y_{p,1}(t) = t^s Q_0(t) e^t.$$

Since $\gamma = 1$ is the root to the characteristic equation we have

$$y_{p,1}(t) = tq_0 e^t, y_{p,1}^{(3)} = 3q_0 e^t + tq_0 e^t.$$

Hence

$$3q_0 e^t = e^t$$

and $q_0 = \frac{1}{3}$. The general solution is

$$y(t) = c_1 e^t + c_2 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) - 7 + \frac{1}{3}e^t.$$

Example 4.

Solve the initial value problem

$$y'' + 4y = 2t, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution. The characteristic equation is

$$r^2 + 4 = 0$$

It has two complex roots $r_{1,2} = \pm 2i$. So the general solution to the homogeneous equation is

$$Y(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

We are looking for a particular solution in the form

$$y_p(t) = t^s Q_1(t) e^{0t}$$

since $\gamma = 0$ is not the root to the characteristic equation $s = 0$ and particular solution has the form

$$y_p(t) = q_0 + q_1 t$$

Obviously

$$y_p''(t) = 0$$

So

$$4q_0 + 4q_1 t = 2t$$

Hence $q_0 = 0$ and $q_1 = \frac{1}{2}$. Hence the general solution is

$$y(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{t}{2}.$$

Using the initial conditions we obtain

$$C_1 = 1, C_2 = \frac{3}{4}.$$

The answer is

$$y(t) = \cos(2t) + \frac{3}{4} \sin(2t) + \frac{t}{2}.$$