LECTURE 11 (267)

Mechanical Vibrations.

Example 1. We consider the mass-spring-dashpot system having the function $x(t)$ and satisfying the equation

$$mx'' + cx' + kx = 0. \tag{1}$$

With the initial condition $x(0) = x_0$ and the initial velocity $x'(0) = v_0$. For the **Critically damped** case show that

$$x(t) = (x_0 + v_0 t + p x_0 t)e^{-pt} \tag{2}$$

Solution. The general solution to equation (1) in the critically damped case is

$$x(t) = (c_1 + c_2 t)e^{-pt}.$$

Obviously

$$x'(t) = -p(c_1 + c_2 t)e^{-pt} + c_2 e^{-pt}.$$

Using the initial condition $x(0) = x_0$ we have

$$x(0) = x_0 = c_1 \tag{3}$$

Hence

$$x'(t) = -p(x_0 + c_2 t)e^{-pt} + c_2 e^{-pt}.$$

Using the initial condition $x'(0) = v_0$ we have

$$x'(0) = v_0 = -p x_0 + c_2$$

Therefore

$$c_2 = v_0 + p x_0. \tag{4}$$

From (3)and (4) formula (2) follows immediately.

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Example 2. We consider the mass-spring-dashpot system having the function $x(t)$ and satisfying the equation

$$mx'' + cx' + kx = 0.$$

With the initial condition $x(0) = x_0$ and the initial velocity $x'(0) = v_0$. For the **Over damped** case show that

$$x(t) = \frac{1}{2\gamma}[(v_0 - r_2 x_0)e^{r_1 t} - (v_0 - r_1 x_0)e^{r_2 t}]. \tag{5}$$

where $\gamma = \frac{1}{2}(r_1 - r_2)$.

Solution. The general solution to equation (1) in overdamped case is

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{6}$$

and

$$x'(t) = r_1 c_1 e^{r_1 t} + r_2 c_2 e^{r_2 t}. \tag{7}$$

Using the initial conditions from (6), (7) we obtain the system of linear equations for unknown coefficients c_1 c_2:

$$\begin{cases} c_1 + c_2 = x_0 \\ r_1 c_1 + r_2 c_2 = v_0 \end{cases}$$

Solving the first equation respect to c_2 we have

$$c_2 = x_0 - c_1.$$

Hence from the second equation we obtain

$$r_1 c_1 + r_2(x_0 - c_1) = v_0.$$

or

$$2\gamma c_1 = v_0 - r_2 x_0.$$

Hence

$$c_1 = \frac{1}{2\gamma}(v_0 - r_2 x_0), \quad c_2 = x_0 - \frac{1}{2\gamma}(v_0 - r_2 x_0) = \frac{1}{2\gamma}(r_1 x_0 - v_0). \tag{8}$$

From (8),(7) follows (5).

Example 3. We consider the mass-spring-dashpot system having the function $x(t)$ and satisfying the equation

$$mx'' + cx' + kx = 0.$$

With the initial condition $x(0) = x_0$ and the initial velocity $x'(0) = v_0$. For the **Under damped** case show that

$$x(t) = e^{-pt}(x_0 cos(\omega_1 t) + \frac{v_0 + px_0}{\omega_1} sin(\omega_1 t)) \tag{9}$$

Solution.

The general solution to equation (1) in the underdamped case is

$$x(t) = e^{-pt}(c_1 cos(\omega_1 t) + c_2 sin(\omega_1 t)) \tag{10}$$

Note that

$$x'(t) = -pe^{-pt}(c_1 cos(\omega_1 t) + c_2 sin(\omega_1 t)) + e^{-pt}(c_1 \omega_1 sin(\omega_1 t) + c_2 \omega_1 cos(\omega_1 t)) \tag{11}$$

Using the initial conditions we obtain from (10),(11)

$$c_1 = x_0, \quad v_0 = -pc_1 + c_2 \omega_1$$

Hence

$$c_1 = x_0, \quad c_2 = \frac{v_0 + px_0}{\omega_1}. \tag{11}$$

From (11), (10) we obtain (9).

Example 4. Let $m = \frac{1}{2}, c = 3, k = 4, x_0 = 2, v_0 = 0$. Find $x(t)$.

Solution. First we find the critical damping $c_{cr} = \sqrt{8} < c = 3$. Hence we have the overdamped case. Note that $p = 3, r_1 = -2, r_2 = -4, \gamma = 1$. Using the formula (5) we have

$$x(t) = \frac{1}{2}(8e^{-2t} - 4e^{-2t}).$$

Example 5. Let $m = 1, c = 8, k = 16, x_0 = 5, v_0 = -10$. Find $x(t)$.

Solution First we find the critical damping $c_{cr} = 8 = c = 3$. Hence we have the critically damped case. Note that $p = 4$. Using the formula (2) we obtain

$$x(t) = (2 - 10t + 20t)e^{-4t}.$$

Example 6. Let $m = 1, c = 10, k = 125, x_0 = 6, v_0 = 50$. Find $x(t)$.

*Solution.*First we find the critical damping $c_{cr} = 10\sqrt{5} > c = 10$. Hence we have the underdamped case. Note that $p = 5$, $\omega_1 = 10$. Using the formula (9) we have

$$x(t) = e^{-5t}(6cos(10t) + \frac{50 + 30}{10} sin(10t)).$$