VECTOR FIELDS

Definition 1. *Let D be a set on the plane. A* **vector field** *on D is a function* $\mathbf{F}$ *that assigns to each point (x, y) in D a two-dimensional vector* $\mathbf{F}(x, y)$.

Denote by $\vec{i}$ the unit vector in direction of the axis x and by $\vec{j}$ the unit vector in direction of the axis y. Denote by $P(x, y)$ the first coordinate of the vector $\mathbf{F}(x, y)$ and by $Q(x, y)$ the second coordinate of the vector $\mathbf{F}(x, y)$. In that case we can represent a vector field F as follows

$$\mathbf{F} = P(x, y)\vec{i} + Q(x, y)\vec{j}. \tag{1}$$

Definition 2. *Let E be a subset on $\mathbb{R}^3$. A* **vector field** *on $\mathbb{R}^3$ is a function* $\mathbf{F}$ *that assigns to each point (x, y, z) in E a three-dimensional vector* $\mathbf{F}(x, y, z)$.

For vector fields on $\mathbb{R}^3$ we have a formula similar to (1)

$$\mathbf{F} = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}. \tag{2}$$

Here $\vec{k}$ is the unit vector in the direction of the axis z and $R(x, y, z)$ is the third coordinate of the vector $\mathbf{F}$.

Definition 3. *We say that the vector field* $\mathbf{F}$ *is conservative if there exists a function $f(x, y, z)$ such that*

$$\mathbf{F} = \nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k}.$$

Typical examples of the vector fields.

Example 1. A fluid occupies the region E in $\mathbb{R}^3$. At an arbitrary point with the coordinates (x, y, z) we measure the velocity of the fluid $\mathbf{V}(x, y, z)$. So we obtain the **Velocity** vector field.

Example 2. If we can measure the gravitation force at each point of universe we obtain the gravitational vector field. In the simplified situation

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when we have only two objects of mass m and M and the object of mass M is located at the origin, the Newton's law of Gravitation provide the formula

$$\mathbf{F}(x,y,z)=\frac{-mMGx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{i}+\frac{-mMGy}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{j}+\frac{-mMGz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{k}.$$

Here G is the gravitational constant.

Example 3. Suppose an electric charge Q is located at the origin. According to Coulomb's law the electric force $\mathbf{F}$ exerted by this charge at charge q located at the point (x,y,z) is

$$\mathbf{F}(x,y,z)=\frac{\epsilon Qqx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{i}+\frac{\epsilon Qqy}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{j}+\frac{\epsilon Qqz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\vec{k}.$$

Example 4. Given a conservative vector field $\mathbf{F}=(2x+y)\vec{i}+(x+2y)\vec{j}$. Find a potential function f.

Solution. We know that there exist a function $f(x,y)$ such that

$$f_x(x,y)=2x+y, \tag{3}$$

$$f_y(x,y)=x+2y. \tag{4}$$

Taking the antiderivative of the right and left hand sides of equation (3) respect to the variable x we obtain

$$f(x,y)=x^2+yx+C_1(y). \tag{5}$$

Here $C_1(y)$ is unknown function which depends on variable y.

Taking the antiderivative of the right and left hand sides of equation (4) respect to the variable y we obtain

$$f(x,y)=y^2+yx+C_2(x). \tag{6}$$

Here $C_2(x)$ is unknown function which depends on variable y. From (5), (6) we obtain

$$x^2+yx+C_1(y)=y^2+yx+C_2(x)$$

or

$$x^2+C_1(y)=y^2+C_2(x). \tag{7}$$

Moving all the terms depending on the variable x in the left hand side and all terms depending on the variable y in the right hand side we have

$$x^2-C_2(x)=-C_1(y)+y^2. \tag{8}$$

Now in the left hand side of (8) we have a function which depends only on the variable x and in the right hand side we have a function which depends only on the variable y. Since x and y are the linearly independent variables this equality is possible if and only if

$$x^2 - C_2(x) = -C_1(y) + y^2 = -C.$$

Thus $C_2(x) = x^2 + C$ and from (6) we obtain

$$f(x, y) = y^2 + yx + x^2 + C.$$