LECTURE 11 (267)

General Solutions of Linear Equations

Example 1. Find the general solution to the ordinary differential equation

$$y^{(4)} - 16y = 0$$

Solution. The characteristic polynomial is

$$r^4 - 16$$

It can be factorized as

$$r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r - 2)(r + 2)(r - 2i)(r + 2i).$$

So we have two real roots $r_{1,2} = \pm 2$ and two complex roots $r_{3,4} = \pm 2i$. Each root has the multiplicity one. So the general solution is

$$Y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 cos(2t) + c_4 sin(2t).$$

Example 2. Find a general solution to equation

$$5y^{(4)} + 3y^{(3)} = 0.$$

Solution. The characteristic polynomial is

$$5r^4 + 3r^3 = 5r^3(r + \frac{3}{5})$$

We have on root $r_1 = 0$ of multiplicity 3 and the root $r_2 = -\frac{3}{5}$ of multiplicity one. The general solution is

$$Y(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-\frac{3}{5}t}.$$

*Example 3.*Find a general solution to equation

$$y^{(3)} + 27y = 0$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Solution. The characteristic polynomial is

$$r^3 + 27$$

It can be factorized as

$$r^3 + 27 = (r+3)(r^2 - 3r + 9) = (r+3)(r - (\frac{3}{2} + \frac{\sqrt{27}}{2})(r - (\frac{3}{2} - \frac{\sqrt{27}}{2})$$

The general solution is

$$Y(t) = c_1 e^{-3t} + c_2 e^{\frac{3}{2}t} cos(\frac{\sqrt{27}}{2}t) + c_2 e^{\frac{3}{2}t} sin(\frac{\sqrt{27}}{2}t).$$

Example 3. Solve the initial value problem

$$y'' - 4y' + 3y = 0, \quad y(0) = 7,\ y'(0) = 11.$$

Solution. First we find a general solution to this equation. The characteristic polynomial is

$$r^2 - 4r + 3 = (r-1)(r-3)$$

Hence the general solution is

$$Y(t) = c_1 e^t + c_2 e^{3t}.$$

The derivative of this function is

$$Y'(t) = c_1 e^t + 3c_2 e^{3t}$$

Using the initial conditions we obtain the system of linear equations

$$c_1 + c_2 = 7,\ c_1 + 3c_2 = 11.$$

Hence $c_2 = 2$ and $c_1 = 5$. So the solution to the initial value problem is

$$y(t) = 5e^t + 2e^{3t}.$$

Example 4. Solve the initial value problem

$$y^{(3)} + 10y'' + 25y', \quad y(0) = 3,\ y'(0) = 4,\ y''(0) = 5$$

*Solution.*The characteristic equation is

$$r^3 + 10r^2 + 25r = r(r+5)^2$$

The root $r_1 = 0$ has the multiplicity one and the root $r_2 = -5$ has the multiplicity two.

So the general solution is

$$Y(t) = c_1 + c_2 e^{-5t} + c_3 t e^{-5t}.$$

The first and the second derivatives of this solution are

$$Y'(t) = -5c_2 e^{-5t} + c_3 e^{-5t} - 5c_3 t e^{-5t}.$$

and

$$Y''(t) = 25c_2 e^{-5t} - 10c_3 e^{-5t} + 25c_3 t e^{-5t}.$$

Using the initial conditions we obtain the system of linear equations

$$c_1 + c_2 = 3 \quad -5c_2 + c_3 = 4, \quad 25c_2 - 10c_3 = 5$$

Hence $c_3 = -3$ and $c_2 = -\frac{7}{5}$ and $c_1 = \frac{22}{5}$. Hence the solution is

$$y(t) = \frac{22}{5} - \frac{7}{2} e^{-5t} - 3te^{-5t}.$$