TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

Today we learn how to use the Cylindrical and Spherical cooordinates for evaluation of the triple integrals. Suppose that the solid occupies the region E and E is the region of the type **I**.

$$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \le z \le \phi_2(x, y)\}.$$

Suppose we can describe the region D as

$$D = \{(r, \theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}.$$

Then in the polar coordinate system we have

$$E = \{(r, \theta, z) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta),\ \ \phi_1(x, y) \le z \le \phi_2(x, y)\}$$

$$\int\int\int_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{\phi_1(rcos\theta, r\sin\theta)}^{\phi_2(rcos\theta, r\sin\theta)} f(rcos(\theta), rsin(\theta), z) r dz dr d\theta. \tag{1}$$

For spherical coordinates we have

$$\int\int\int_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho sin\phi cos\theta, \rho sin\phi sin\theta, \rho cos\theta) \rho^2 sin\phi d\rho d\theta d\phi, \tag{2}$$

where E is given by

$$E = \{(\rho, \theta, \phi) | a \le \rho \le b,\ \ \alpha \le \theta \le \beta,\ \ c \le \phi \le d\}.$$

Example 1. Evaluate $\int\int\int_E (x^2 + y^2) dV$, where E is the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = -1$ and $z = 2$.

Solution We can describe our region E in cylindrical coordinate system as

$$E = \{(r, \theta, z) | 0 \le r \le 2,\ \ 0 \le \theta < 2\pi,\ \ -1 \le z \le 2\}.$$

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Hence using the formula (1) we have

$$\int\int\int_E (x^2+y^2)dV = \int_0^{2\pi}\int_0^2\int_{-1}^2 (r^2cos^2\theta + r^2sin^2\theta)rdzdrd\theta =$$

$$\int_0^{2\pi}\int_0^2\int_{-1}^2 r^3dzdrd\theta = \int_0^{2\pi}\int_0^2 zr^3|_{-1}^2 drd\theta = 3\int_0^{2\pi}\int_0^2 r^3drd\theta = 3\int_0^{2\pi}\frac{r^4}{4}|_0^2 d\theta =$$

$$12\int_0^{2\pi} 1d\theta = 24\pi.$$

Example 2. Evaluate $\int\int\int_E x^2dV$, where E is the solid that lies within the cylinder $x^2+y^2=1$ above the plane $z=0$, and below the cone $z^2=4x^2+4y^2$.

Solution. The equation for the cone $z^2=4x^2+4y^2$ in cylindrical coordinate system is $z^2=4r^2$. Since we are interesting in the part of the cone which is above the plane $z=0$ we can rewrite the equation for cone as

$$z=2r.$$

Therefore we can describe the set E as

$$E=\{(r,\theta,z)|0\le r\le 1,\ 0\le\theta<2\pi,\ 0\le z\le 2r\}.$$

Hence using the formula (1) we have

$$\int\int\int_E x^2dV = \int_0^{2\pi}\int_0^1\int_0^{2r}(r^2cos^2\theta)rdzdrd\theta =$$

$$\int_0^{2\pi}\int_0^1 zcos^2\theta r^3|_0^{2r}drd\theta = \int_0^{2\pi}\int_0^1 cos^2\theta 2r^4drd\theta = \int_0^{2\pi} cos^2\theta 2\frac{r^5}{5}|_0^{`} d\theta =$$

$$\frac{1}{5}\int_0^{2\pi} cos^2\theta d\theta.$$

Note that

$$\int cos^2udu = \frac{1}{2}u + \frac{1}{4}sin2u.$$

So

$$\int\int\int_E x^2dV = \frac{1}{5}(\frac{1}{2}\theta + \frac{1}{4}sin2\theta)|_0^{2\pi} = \frac{1}{5}\pi.$$

Example 3. Evaluate $\int\int\int_B (x^2+y^2+z^2)dV$, where B is the unit ball $x^2+y^2+z^2\le 1$.

Solution. In the spherical coordinate system one can describe the ball as

$$B = \{(\rho, \theta, \phi) | 0 \leq \rho \leq 1, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi\}.$$

Also we note that $\rho^2 = x^2 + y^2 + z^2$. Hence using the formula (2) we have

$$\int\int\int_B (x^2+y^2+z^2) dV = \int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^4 sin\phi d\phi d\theta d\rho = -\int_0^1 \int_0^{2\pi} \rho^4 cos\phi|_0^{\pi} d\theta d\rho$$

$$= 2\int_0^1 \int_0^{2\pi} \rho^4 d\theta d\rho = 2\int_0^1 z\rho^4|_0^{2\pi} d\rho$$

$$= 4\pi \int_0^1 \rho^4 d\rho = \frac{4\pi}{5}\rho^5|_0^1 = \frac{4\pi}{5}.$$