LECTURE 10 (267)

General Solutions of Linear Equations

The general linear ordinary differential equation of order n has the form

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \ldots P_{n-1}(t)y' + P_n(t)y = F(t). \tag{1}$$

If we assume that $P_0(t) \neq 0$ on the interval I we can reduce this equation to the form

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots p_{n-1}(t)y' + p_n(t)y = f(t). \tag{2}$$

If the function $f \equiv 0$ we say that the equation (2) is homogeneous:

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots p_{n-1}(t)y' + p_n(t)y = 0. \tag{3}$$

Theorem 1. *Let $y_1, \ldots, y_n$ be n solutions of the homogeneous linear equation (3) on the interval I. If $c_1, \ldots, c_n$ are constants then the linear combination*

$$y(t) = c_1y_1(t) + c_2y_2 + ... + c_ny_n$$

is also solution to equation (3).

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The following theorem provide the existence and uniqueness result for the initial value problem associated with the equation (2).

Theorem 2. *Suppose that the functions $p_1, p_1, \ldots, p_n$ and f are continuous on the open interval I containing the ppoint a. Then, given n numbers $b_0, b_1, ..., b_{n-1}$ the nth-order linear equation (2)*

$$y^{(n)} + p_1(t)y^{(n-1)} + \ldots p_{n-1}y' + p_n(t)y = f(t).$$

has a unique solution on the interval I that satisfies the n initial conditions

$$y(a) = b_0,\ y'(a) = b_1,\ \ldots y^{(n-1)}(a) = b_{n-1} \tag{4}$$

Definition. The n functions $f_1, f_2, \ldots, f_n$ are said to be linearly dependent on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1 + c_2 f_2 + \ldots c_n f_n = 0$$

on I.

Suppose that $y_1, \ldots, y_n$ are solutions to the homogeneous equation (3). If they are linearly independent then the general solution to equation (3) is given by formula:

$$Y(t) = c_1 y_1(t) + c_2 y_2 + \ldots.. + c_n y_n$$

Denote by y_p some solution to equation (2). Let $y_1, \ldots, y_n$ are linearly independent solutions to homogeneous equation (3) . Then the general solution to equation (3) is

$$y(t) = c_1 y_1(t) + c_2 y_2 + \ldots.. + c_n y_n + y_p(t).$$

Example 1. Find a particular solution satisfying the given initial conditions

$$y^{(3)} + 2y'' - y' - 2y = 0, y(0) = 1,\ y'(0) = 2,\ y''(0) = 0,\ y_1 = e^t,\ y_2 = e^{-t},\ y_3 = e^{-2t}.$$

Solution. the general solution to the ordinary differential equation is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}$$

Let's find the first and the second order derivatives of this solution

$$y'(t) = c_1 e^t - c_2 e^{-t} - 2c_3 e^{-2t}$$

and

$$y''(t) = c_1 e^t + c_2 e^t + 4c_3 e^{-2t}$$

We have

$$\begin{cases} y(0) = 1 = c_1 + c_2 + c_3 \\ y'(0) = 2 = c_1 - c_2 - 2c_3 \\ y''(0) = 0 = c_1 + c_2 + 4c_3 \end{cases}$$

One can rewrite this system in the form

$$\begin{cases} c_1 + c_2 + c_3 = 1 \\ c_1 - c_2 - 2c_3 = 2 \\ c_1 + c_2 + 4c_3 = 0 \end{cases}$$

We subtract from the second and third equations the first one

$$\begin{cases} c_1 + c_2 + c_3 = 1 \\ -2c_2 - 3c_3 = 1 \\ 3c_3 = -1 \end{cases}$$

Therefore

$$\begin{cases} c_1 + c_2 - \frac{1}{3} = 1 \\ -2c_2 = 0 \\ c_3 = -\frac{1}{3} \end{cases}$$

Finally we have

$$\begin{cases} c_1 = \frac{4}{3} \\ -2c_2 = 0 \\ c_3 = -\frac{1}{3} \end{cases}$$

The solution to the initial value problem is

$$y(t) = \frac{4}{3}e^t - \frac{1}{3}c_3e^{-2t}$$

Example 2. Find a solution satisfying the given initial condition

$$y'' + y = 3t,\ y(0) = 2,\ y'(0) = -2,\ y_p = 3t.$$

Solution. The general solution to the homogeneous ordinary differential equation $y'' + y = 0$ is

$$Y(t) = c_1cost + c_2sint.$$

Hence the general solution to our equation is

$$y(t) = 3t + c_1cost + c_2sint.$$

Taking the derivative of the function $y'(t)$ we obtain

$$y'(t) = 3 - c_1sint + c_2cost.$$

Then

$$y(0) = 2 = c_1 \quad and\ y'(0) = -2 = 3 + c_2$$

from the first equation we have

$$c_1 = 2$$

and from the second one

$$c_2 = -5.$$

Hence the solution is

$$y(t) = 3t + 2cost - 5sint.$$

Example 3. Use the Wronskian to prove that functions $f(x) = e^x, g(x) = e^{2x}, h(x) = e^{3x}$ are linearly independent on the real line.

Solution.

$$W(f,g,h) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x}(18+3+4-2-12-9) = 2e^{6x} \neq 0$$