TANGENT PLANES AND DIFFERENTIALS

Suppose we have some surface S in three dimensional space.There are three way to describe this object mathematically.

1. A surface is given as a graph of function: $z = f(x, y)$ where $(x, y) \in D$ and D is a region on the $xy - plane$. In that case a point $P = (x_0, y_0, z_0)$ is on the surface S if and only if $(x_0, y_0) \in D$ and $z_0 = f(x_0, y_0)$.

2. A surface given by equation $F(x, y, z) = 0$. In that case a point $P = (x_0, y_0, z_0)$ is on the surface S if and only if $F(x_0, y_0, z_0) = 0$.

3.A surfice is given in parametric form. It is so called *parametric surface.* Such surfaces we'll study later in Chapter 14. Let us remind that the general equation of a plane is

$$Ax + By + Cz + D = 0.$$

Here A, B, C, D are some fixed constants.

Defenition 1.. *An equation of the tangent plane to the surface* $z = f(x, y)$ *at the point* $P = (x_0, y_0, z_0)$ *is*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \quad (1)$$

Defenition 2.. *An equation of the tangent plane to the surface* $F(x, y, z) = 0$ *at the point* $P = (x_0, y_0, z_0)$ *is*

$$F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0. \quad (2)$$

Example 1. Find an equation of the tangent plane to a given surface at the specified point: $z = x^2 + 4y^2$ and $P = (2, 1, 8)$.

Solution. The surface is given as a graph of function so we use the equation (1). In our case $f(x, y) = x^2 + 4y^2$. Then $f_x(x, y) = 2x$ and $f_y(x, y) = 8y$. Obviously

$$f_x(2, 1) = 4, f_y(2, 1) = 8.$$

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Applying the formula (1) we obtain

$$z - 8 = 4(x - 2) + (y - 1).$$

Example 2. Find an equation of the tangent plane to a given surface at the specified point: $z = x^2 - y^2$ and $P = (3, -2, 5)$.

Solution. The surface is given as a graph of function so we use the equation (1). In our case $f(x, y) = x^2 - y^2, f_x(x, y) = 2x$ and $f_y(x, y) = -2y$. Obviously

$$f_x(3, -2) = 6, \quad f_y(3, -2) = 4.$$

Applying the formula (1) we obtain

$$z - 5 = 6(x - 3) + 4(y + 2).$$

Example 3. Find an equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point $P = (x_0, y_0, z_0)$.

Solution. The surface is given by equation $F(x, y, z) = 0$ with

$$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

Obviously

$$F_x(x_0, y_0, z_0) = 2\frac{x_0}{a^2}, \quad F_y(x_0, y_0, z_0) = 2\frac{y_0}{b^2}, \quad F_z(x_0, y_0, z_0) = 2\frac{z_0}{c^2}.$$

Therefore according to formula (2) we have

$$2\frac{x_0}{a^2}(x - x_0) + 2\frac{y_0}{b^2}(y - y_0) + 2\frac{z_0}{c^2}(z - z_0) = 0. \tag{3}$$

We can divide equation (3) by the constant 2 and rewrite in slightly different way:

$$\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z + \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 0. \tag{4}$$

Since the point $P = (x_0, y_0, z_0)$ is on the ellipsoid we have

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1. \tag{5}$$

Using the identity (5) one can transform equation (4) to the form

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} + 1 = 0. \tag{6}$$

Example 4. On the sphere $x^2 + y^2 + z^2 = 1$ find all points such that tangent planes to the sphere at these points are parallel to the plane $x = 3$.

Solution. Two planes $Ax+By+Cz+D = 0$ and $A_1x+B_1y+C_1z+D_1 = 0$ are paralllel if and only if there exist a number α such that

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \alpha \begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix}. \tag{7}$$

Let a point $P = (x_0, y_0, z_0)$ be an arbitrary point on the sphere. Then the tangent plane to the sphere at this point is

$$xx_0 + yy_0 + zz_0 + 1 = 0.$$

Here we used the formula (6) from the previous example.

In our case equation (7) has the form

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{8}$$

From equation (8) we obtain $y_0 = z_0 = 0$. On the other hand the point P is on the sphere. Hence

$$x_0^2 + y_0^2 + z_0^2 = 1.$$

From this equation, since $y_0 = z_0 = 0$ we obtain

$$x_0^2 = 1.$$

So $x_0 = \pm 1$. Thus we obtain two points $(1, 0, 0)$ and $(-1, 0, 0)$.