TRIPLE INTEGRALS

In the three dimensional space we introduce a rectangular box

$$B = \{(x, y, z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}.$$

Here a, b, c, d, r, s are an arbitrary but fixed numbers. the first step is to partition the intervals $[a, b]$, $[c, d]$, and $[r, s]$ as follows

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i \dots x_l = b,$$

$$c = y_0 < y_1 < \cdots < y_{j-1} < y_j \dots y_m = d$$

$$r = z_0 < Z_1 < \cdots < z_{k-1} < z_k < \dots z_n = s.$$

The planes through these partition points parallel to the coordinate planes divide the box B into ℓmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k].$$

The volume of B_{ijk} is

$$\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k,$$

where

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1}, \quad \Delta z_k = z_k - z_{k-1}.$$

Next we form the Riemann sum

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk},$$

where a point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in rectangular box B_{ijk}.

Denote by $\|P\|$ the length of the longest diagonal of all boxes B_{ijk}.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition 1. *The triple integral of the function f over the box B is*

$$\int\int\int_B f(x,y,z)dV = \lim_{\|P\|\to 0} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V_{ijk}.$$

if this limit exists.

For the triple integral over rectangular box we have the following Theorem.

Fubini's Theorem for triple integrals. *If the function f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$ then*

$$\int\int\int_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz.$$

Definition 2. *We say that E is the region of the type* **I** *if it can be represented in the form*

$$E = \{(x,y,z)|(x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\},$$

where D is some region on the plane.

If E region of the type **I** we have

$$\int\int\int_E f(x,y,z)dV = \int\int_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dz\right] dA.$$

If D is the region of type **I** we can rewrite the previous formula as

$$\int\int\int_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dzdydx.$$

If D is the region of type **II** we can rewrite the previous formula as

$$\int\int\int_E f(x,y,z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dzdxdy.$$

If we have a solid which occupies the region E then the volume of this solid $V(E)$ is given by triple integral

$$V(E) = \int\int\int_E 1dV.$$

The mass m of a solid which occupies the region E with the density function $\rho(x,y,z)$ is given by formula

$$m = \int\int\int_E \rho(x,y,z)dV.$$

The **center of mass** located at the point $(\bar{x},\bar{y},\bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xz}}{m},$$

$$M_{yz} = \int\int\int_E x\rho(x,y,z)dV, \quad M_{xz} = \int\int\int_E y\rho(x,y,z)dV, \quad M_{xz} = \int\int\int_E z\rho(x,y,z)dV.$$

The **moments of inertia** are

$$I_x = \int\int\int_E (y^2+z^2)\rho(x,y,z)dV, \quad I_y = \int\int\int_E (x^2+z^2)\rho(x,y,z)dV,$$
$$I_z = \int\int\int_E (x^2+y^2)\rho(x,y,z)dV.$$

Example 1. Evaluate the iterated integral

$$\int_0^1 \int_0^z \int_0^y xyzdxdydz.$$

Solution.

$$\int_0^1 \int_0^z \int_0^y xyzdxdydz = \int_0^1 \int_0^z \frac{x^2yz}{2}|_0^y dydz = \int_0^1 \int_0^z \frac{y^3z}{2}dydz = \int_0 61 \int_0^z \frac{y^3z}{2}dydz$$
$$= \int_0^1 \frac{y^4z}{8}|_0^z dz = \int_0^1 \frac{z^5}{8}dz = \frac{z^6}{48}|_0^1 = \frac{1}{48}.$$

Example 2. Evaluate the triple integral

$$\int\int\int_E yzdV,$$

where

$$E = \{(x,y,z)|0 \le z \le 1, 0 \le y \le 2z, 0 \le x \le z+2\}.$$

Solution. Our double integral equals to the iterated integral

$$\int\int\int_E yzdV = \int_0^1\int_0^{2z}\int_0^{z+2} yzdxdydz = \int_0^1\int_0^{2z} xyz|_0^{z+2}dydz$$

$$\int_0^1\int_0^{2z}(z+2)yzdydz = \int_0^1\int_0^{2z}(z^2+2z)ydydz$$

$$= \int_0^1 (z^2+2z)\frac{y^2}{2}\int_0^{2z} dz = 2\int_0^1 (z^2+2z)z^2dz = 2\int_0^1 (z^4+2z^3)dz = 2(\frac{z^5}{5}+\frac{z^4}{2})|_0^1$$

$$= 2(\frac{1}{5}+\frac{1}{2}).$$

Example 3. Find the volume of a solid enclosed by paraboloids $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$.

Solution.

$$V(E) = \int\int\int_E 1dv = \int\int_D\int_{x^2+y^2}^{18-x^2-y^2} 1dzdA = \int\int_D z|_{x^2+y^2}^{18-x^2-y^2} dA$$

$$= \int\int_D (18 - 2(x^2+y^2))dA =$$

$$\int_0^{2\pi}\int_0^3 (-2r^2+18)rdrd\theta = \int_0^{2\pi}(-\frac{r^4}{2}+9r^2)|_0^3 d\theta = \int_0^{2\pi}\frac{3^4}{2}d\theta = 3^4\pi.$$