LECTURE 1 (267)

In this course we are going to study the **ordinary differential equations**. What is the ordinary differential equation? Denote by $y(t)$ be a smooth function with the domain $[a, b]$, and the function $f(t, z)$ be a function of two variables. Then the particular example of an ordinary differential equation is

$$\frac{dy}{dt} = f(t, y(t)) \quad t \in [a, b] \tag{1}$$

How to understand a equation (1)? The unknown is the function $y(t)$. So we are looking for a function $y(t)$ such that for any t from a segment $[a, b]$ we have the equality $\frac{dy}{dt} = f(t, y(t))$. We note that in general solution to equation (1) is not unique. Let us consider the specific example. Let $f(t, z) = z$. Then

$$\frac{dy}{dt} = y(t). \tag{2}$$

The general solution to this ordinary differential equation is

$$y(t) = ae^t.$$

Here a is an arbitrary constant. Say the function $y(t) = e^t$ is a solution to equation (2) and the function $y(t) = 3e^t$ is solution also. We are already mentioned the fact that a solution to the ordinary differential equation is not unique.

In fact we are already familiar with the simplest types of ordinary differential equations. The operation of taking antiderivative. We remind that if $f(t)$ is a function than antiderivative of this function is $F(t)$ such that $F'(t) = f(t)$. We can rewrite this operation using the ordinary differential equation

$$\frac{dy}{dt} = f(t)$$

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In that case the solution to this ordinary differential equation a function $y(t)$ is the antiderivative of the function $f(t)$. The general solution to the ordinary differential equation given by formula

$$y(t) = \int_a^t f(s)ds + C.$$

Now we introduce the definition of the general ordinary differential equation. Let function $F(t, z_0, z_1, ..., z_n)$ be a function of $n + 2$ variables, for a function $y(t)$ we denote the derivative of an order n as $y^{(n)}$. Then the form of general ordinary differential equation is

$$F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0. \tag{3}$$

Classification of the ordinary differential equations.

We say that ordinary differential equation (3) is **linear ordinary differential equation** if the function F is the linear function of the variables $z_0, z_1, ..., z_n$. If the ordinary differential equation is not linear we say that it is **nonlinear ordinary differential equation.** For example the equation $y'(t)+2y(t)+sin(t) = 0$ is the linear equation. Really it can be written in the form of (3) with the function $F(t, z_0, z_1) = z_1 + 2z_0 + sin(t)$. In variables z_1 and z_2 this function is linear. So the ordinary differential equation is linear. Om the other hand the equation

$$y'(t) + y^2 = 1 \qquad ((4))$$

is the nonlinear ordinary differential equation. Really the function $F(t, z_0, z_1)$ is given by formula

$$F(t, z_0, z_1) = z_1 + z_0^2 - 1.$$

This function is nonlinear in variable z_0, so the ordinary differential equation is nonlinear.

By **order** of the ordinary differential equation we mean the order of the highest derivative that appears in the left hand side of (3). For example the equations given by formulas (1), (2), (4) are the first order ordinary differential equations. The equation

$$(y^{(3)}(t))^4 + y(t) = cos(t).$$

is the third order differential equation.

Initial value problem

by the initial value problem for a first order ordinary differential equation we mean the following problem. Let the time moment t_0 and the number y_0 are fixed. We are looking for solution of the ordinary differential equation

$$\frac{dy}{dt} = f(t, y) \tag{5}$$

such that

$$y(t_0) = y_0 \tag{6}$$

For the second order ordinary differential equation the initial value problem can be written as follows:

$$\frac{d^2y}{dt^2} = f(t, y, y') \tag{7}$$

such that

$$y(t_0) = y_0, \quad y_t(t_0) = y_1 \tag{8}$$

Here the numbers y_0, y_1 are given.

Example 1. Let us consider an object moving along the line with constant speed v_0. This is the simplest example of motion. We introduce the coordinate system on this line. Then $x(t)$ is position of an object at moment t. Velocity of an object is $\frac{dx}{dt}$.Assume that the "direction" of the coordinate system match the direction of motion. Then the function $x(t)$ satisfies the ordinary differential equation

$$\frac{dx}{dt} = v_0 \tag{9}$$

If we know the position of an object at moment t_0- say $x(t_0) = x_0$ then we have the initial value problem

$$\frac{dx}{dt} = v_0, \quad x(t_0) = x_0 \tag{10}$$

Then the general solution to equation (9) is

$$x(t) = \int_{t_0}^{t} v ds + C = (t - t_0)v + C. \tag{11}$$

Now among the set of general solutions we should find one which satisfies the condition $x(t_0) = x_0$. Really we plug in formula (11) t_0 instead of t then

$$x(t_0) = x_0 = (t_0 - t_0)v + C$$

Hence $C = x_0$ and we arrive to thew formula

$$x(t) = (t - t_0)v + x_0. \tag{12}$$