## IMPORTANT FORMULAS FOR THE CHAPTER 13.

$$R = \{(x, y)|a \le x \le b, c \le y \le d\}.$$

**Fubini's Theorem.** *If $f$ is continuous function on the rectangle $R = \{(x, y)|a \le x \le b, c \le y \le d\}$ then*

$$\int\int_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy.$$

*More generally, this is true if we assume that $f$ is bounded on $R$, $f$ is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.*

**Polar coordinates.**

$$x = rcos\theta, \quad y = rsin\theta.$$

**Definition.** *By polar rectangle $R$ we mean the following region on the plane*

$$R = \{(r, \theta)|a \le r \le b, \alpha \le \theta \le \beta\}.$$

Suppose that $f$ is continuous function on a polar rectangle $R$ given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$

$$\int\int_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(rcos\theta, rsin\theta)rdrd\theta.$$

If the region $D$ on a plane can be represented in the form

$$D = \{(r, \theta)|\alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}.$$

then

$$\int\int_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(rcos\theta, rsin\theta)rdrd\theta.$$

The *moment* of the lamina about the $x - axis$ is given by formula

$$M_x = \int\int_D y\rho(x,y)dA.$$

and the moment about the $y - axis$ is

$$M_y = \int\int_D x\rho(x,y)dA.$$

**Definition.** *The coordinates* $(\overline{x}, \overline{y})$ *of the center of mass of a lamina occupying the region* $D$ *and having density function*$\rho(x,y)$ *are*

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\int\int_D x\rho(x,y)dA, \quad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\int\int_D y\rho(x,y)dA.$$

The moment of inertia about the $x - axis$ is

$$I_x = \int\int_D y^2\rho(x,y)dA.$$

The moment of inertia about the $y - axis$ is

$$I_y = \int\int_D x^2\rho(x,y)dA.$$

The polar moment of inertia is

$$I_0 = \int\int_D (x^2 + y^2)dA.$$

**Theorem.** *The area of the surface given as a graph of the function* $z = f(x,y)$ *over the region* $(x,y) \in D$ *is*

$$A(S) = \int\int_D \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}dA$$

*provided that the functions* $f_x$ *and* $f_y$ *are continuous over the region* $D$.

$$B = \{(x,y,z)|a \le x \le b, c \le y \le d, r \le z \le s\}.$$

**Fubini's Theorem for triple integrals.** *If the function $f$ is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$ then*

$$\int\int\int_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz.$$

**Definition 2.** *We say that $E$ is the region of the type* **I** *if it can be represented in the form*

$$E = \{(x,y,z)|(x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\},$$

*where $D$ is some region on the plane.*

If $E$ region of the type **I** we have

$$\int\int\int_E f(x,y,z)dV = \int\int_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dz\right] dA.$$

If $D$ is the region of type **I** we can rewrite the previous formula as

$$\int\int\int_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dzdydx.$$

If $D$ is the region of type **II** we can rewrite the previous formula as

$$\int\int\int_E f(x,y,z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)dzdxdy.$$

If we have a solid which occupies the region $E$ then the volume of this solid $V(E)$ is given by triple integral

$$V(E) = \int\int\int_E 1dV.$$

The mass $m$ of a solid which occupies the region $E$ with the density function $\rho(x,y,z)$ is given by formula

$$m = \int\int\int_E \rho(x,y,z)dV.$$

The **center of mass** located at the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xz}}{m},$$

$$M_{yz} = \int\int\int_E x\rho(x,y,z)dV, \quad M_{xz} = \int\int\int_E y\rho(x,y,z)dV, \quad M_{xz} = \int\int\int_E z\rho(x,y,z)dV.$$

The **moments of inertia** are

$$I_x = \int\int\int_E (y^2+z^2)\rho(x,y,z)dV, \quad I_y = \int\int\int_E (x^2+z^2)\rho(x,y,z)dV,$$

$$I_z = \int\int\int_E (x^2+y^2)\rho(x,y,z)dV.$$

**Cylindrical coordinates.**

$$\begin{cases} x = rcos\theta \\ y = rsin\theta \\ z = z \end{cases}$$

**Spherical coordinates.**

$$\begin{cases} x = \rho sin\phi cos\theta \\ y = \rho sin\phi sin\theta \\ z = \rho cos\phi \end{cases}$$

Triple integrals in Cylindrical and Spherical coordinates.

$$E = \{(x,y,z) | (x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\}.$$

$$D = \{(r,\theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}.$$

Then

$$\int\int\int_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{\phi_1(rcos\theta, r\sin\theta)}^{\phi_2(rcos\theta, r\sin\theta)} f(rcos(\theta), rsin(\theta), z) rdzdrd\theta.$$

For spherical coordinates we have

$$\int\int\int_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho sin\phi cos\theta, \rho sin\phi sin\theta, \rho cos\theta)\rho^2 sin\phi d\rho d\theta d\phi,$$

where $E$ is given by

$$E = \{(\rho,\theta,\phi) | a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}.$$