ROOT SPACE DECOMPOSITION OF \( g_2 \) FROM OCTONIONS

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ABSTRACT. We describe a simple way to write down explicit derivations of octonions that form a Chevalley basis of \( g_2 \). This uses the description of octonions as a twisted group algebra of the finite field \( F_8 \). Generators of \( \text{Gal}(F_8/F_2) \) act on the roots as 120-degree rotations and complex conjugation acts as negation.

1. Introduction. Let \( O \) be the unique real nonassociative eight dimensional division algebra of octonions. It is well known that the Lie algebra of derivations \( \text{Der}(O) \) is the compact real form of the Lie algebra of type \( G_2 \). Complexifying we get an identification of \( \text{Der}(O) \otimes \mathbb{C} \) with the complex simple Lie algebra \( g_2 \). The purpose of this short note is to make this identification transparent by writing down simple formulas for a set of derivations of \( O \otimes \mathbb{C} \) that form a Chevalley basis of \( g_2 \) (see Theorem 11). This gives a quick construction of \( g_2 \) acting on \( \text{Im}(O) \otimes \mathbb{C} \) because the root space decomposition is visible from the definition. The highest weight vectors of finite dimensional irreducible representations of \( g_2 \) can also be easily described in these terms.

Wilson [W] gives an elementary construction of the compact real form of \( g_2 \) with visible \( 2^3 \cdot L_3(2) \) symmetry. This note started as a reworking of that paper in light of the definition of \( O \) from [B], namely, that \( O \) can be defined as the real algebra with basis \( \{ e^x : x \in F_8 \} \), with multiplication defined by

\[
e^x e^y = (-1)^{\varphi(x,y)} e^{x+y} \quad \text{where} \quad \varphi(x,y) = \text{tr}(yx^6) \tag{1}
\]

and \( \text{tr} : F_8 \to F_2 \) is the trace map: \( x \mapsto x + x^2 + x^4 \). Notice that if \( x \in F_8^* = F_8 - \{0\} \), then in the above formula \( \varphi(x,y) = \text{tr}(yx^{-1}) \).

The definition of \( O \) given above has a visible order–three symmetry \( Fr \) corresponding to the Frobenius automorphism \( x \mapsto x^2 \) generating \( \text{Gal}(F_8/F_2) \), and a visible order–seven symmetry \( M \) corresponding to multiplication by a generator of \( F_8^* \). Together they generate a group of order 21 that acts simply transitively on the natural basis \( B = \{ e^x \land e^y : x, y \in F_8^*, x \neq y \} \) of \( \land^2 \text{Im}(O) \). The only element of \( F_8^* \) fixed by \( Fr \) is 1. Let \( \{0, 1, x, y\} \subseteq F_8 \) be a subset corresponding to any line of \( P^2(F_2) \) containing 1. Let \( B_0 \subseteq B \) be the Frobenius orbit of \( e^x \land e^y \) and let \( B_0, B_1, \ldots, B_6 \) be the seven translates of \( B_0 \) by the cyclic group \( \langle M \rangle \). Then \( B \) is the disjoint union of \( B_0, \ldots, B_6 \).

Using the well known natural surjection \( D : \land^2 \text{Im}(O) \to \text{Der}(O) \) (see definition 2 below), we get a generating set \( D(B) \) of \( \text{Der}(O) \). The kernel of \( D \) has dimension seven with a basis \( \{ \sum_{i \in B_i} b_i : i = 0, \ldots, 6 \} \). The images of \( B_0, \ldots, B_6 \) span seven mutually orthogonal Cartan subalgebras transitively permuted by \( \langle M \rangle \) and forming

\textbf{Date}: July 20, 2017.

\textbf{2010 Mathematics Subject Classification}. Primary: 16W25; Secondary: 16W10, 17B25.

\textbf{Key words and phrases}. Exceptional Lie algebras, Chevalley basis, Octonions, derivations.
an orthogonal decomposition of $\text{Der}(\mathbb{O})$ in the terminology of [KT]. We fix the cartan subalgebra spanned by $D(B_0)$ because it is stable under the action of $\text{Fr}$. The short coroots in this Cartan are $\{ \pm D(b) : b \in B_0 \}$. At this point, it is easy to write down explicit derivations of $\mathbb{O} \otimes \mathbb{C}$ corresponding to a Chevalley basis of $\mathfrak{g}_2$ by simultaneously diagonalizing the action of the coroots (see the discussion preceding Theorem 11).

The following symmetry considerations make our job easy. We choose a root system for $\mathfrak{g}_2$ such that the automorphism $\text{Fr}$ acts on the roots as $120$–degree rotation and the complex conjugation (of $\mathbb{O} \otimes \mathbb{C}$) acts as negation. Together, these automorphisms generate a $\mathbb{Z}/6\mathbb{Z}$ that acts transitively on the short roots (and the long roots) of $\mathfrak{g}_2$. In our construction, the action of this $\mathbb{Z}/6\mathbb{Z}$ is a-priori visible. There are other constructions of $\mathfrak{g}_2$ from split octonions known in the literature (e.g., see [KT], pp. 104–106 or [EK]), but the definition in (1) and the visible $\mathbb{Z}/6\mathbb{Z}$ symmetry makes this approach cleaner and mostly computation free.

2. Definition. Let $a, b \in \mathbb{O}$. Write $\text{ad}_a(b) = [a, b] = ab - ba$. Define $D(a, b) : \mathbb{O} \to \mathbb{O}$ by

$$D(a, b) = \frac{1}{2}(\text{ad}_a + \text{ad}_b).$$

Clearly $D(a, b) = -D(b, a)$, $D(a, b)1 = 0$ and $D(1, a) = 0$. So $D$ defines a linear map from $\wedge^2 \text{Im}(\mathbb{O})$ to $\text{End}(\mathbb{O})$ which we also denote by $D$. So $D(a, b) = D(a \wedge b)$.

Notation: From here on, we shall write $\mathfrak{g} = \text{Der}(\mathbb{O}) \otimes \mathbb{C}$. The Frobenius automorphism $\text{Fr}$ acts on $\mathbb{O}$ and hence on $\mathfrak{g}$. We shall see that $\text{Fr}$ also acts on the roots and coroots of $\mathfrak{g}$, once we fix an appropriate Cartan subalgebra. If $x$ is an element of any of these sets, we sometimes write $x'$ for its image under $\text{Fr}$. Choose $\alpha \in \mathbb{F}_8$ such that $\alpha^3 = \alpha + 1$. Write

$$e_i = e_i^{\alpha^i} \text{ and } e_{ij} = D(e_i \wedge e_j).$$

Note that $\text{Fr} : e_i \mapsto e_{2i}$, that is, $e_i' = e_{2i}$, where the subscripts are read modulo 7.

3. Lemma. Let $x$ and $y$ be distinct elements of $\mathbb{F}_8^*$ and $z \in \mathbb{F}_8$. Then

$$D(e^x \wedge e^y)e^z = \begin{cases} 2e^y & \text{if } z = x, \\ -2e^x & \text{if } z = y, \\ 0 & \text{if } z = 0 \text{ or } z = x + y, \\ -(e^xe^y)e^z & \text{otherwise.} \end{cases}$$

Proof. Let $a, b \in \mathbb{O}$. Define $R(a, b) : \mathbb{O} \to \mathbb{O}$ by $R(a, b) = [\text{ad}_a, \text{ad}_b] - \text{ad}_{[a, b]}$. One verifies that $R(a_1, a_2)(a_3) = -\sum_{\sigma \in S_3} \text{sign}(\sigma)[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}]$ where $[a, b, c] = (ab)c - a(bc)$ is the associator. The properties of the associator in $\mathbb{O}$ implies $R(a_1, a_2)(a_3) = -6[a_1, a_2, a_3]$. So

$$2D(a \wedge b) = \text{ad}_{[a, b]} + \frac{1}{2}R(a, b) = \text{ad}_{[a, b]} - 3[a, b, r].$$

If $z \in \mathbb{F}_2x + \mathbb{F}_2y$, then $e^z$ belongs to the associative subalgebra spanned by $e^x$ and $e^y$ and the Lemma is easily verified in this case. If $z \notin \mathbb{F}_2x + \mathbb{F}_2y$, then using equation (1) one easily verifies that $

\text{ad}_{[e^x, e^y]}e^z = 2[e^x, e^y, e^z].$ The Lemma follows from this and equation (2). \qed

Let $a, b \in \mathbb{O}$. Since the subalgebra of $\mathbb{O}$ generated by $a$ and $b$ is associative, the maps $\text{ad}_{[a, b]}$ and $[\text{ad}_a, \text{ad}_b]$ agree on this subalgebra. Note that the restriction of
2D(a, b) to this subalgebra is just the inner derivation ad_{a, b}. In fact the following
is well known:

4. Lemma. If a, b ∈ O, then D(a ∧ b) is a derivation of O.

By linearity it suffices to show that if x, y are distinct elements of F^2, then
D = D(e^x ∧ e^y) is a derivation of O. Write L(z, w) = D(e^z) e^w + e^z D(e^w) - D(e^z e^w).
It suffices to prove that L(z, w) = 0 for all z, w ∈ F^2. Only a few cases need to be
checked if one first proves the following Lemma.

5. Lemma. (a) Suppose u + x + y and v are distinct elements of F^2. Then D(e^v) and e^v anticommutate.
(b) Suppose u, v and x + y are three distinct elements of F^2. If L(u, v) = 0, then
L(u + v, u) = 0.

One can directly prove Lemmas 4 and 5 using Lemma 3. Since Lemma 4 is well
known (see [S]), we shall omit the details of the proof and move on to describe
the kernel of D : ∧^2 Im(O) → Der(O). Let M : F^2 → F^2 be the automorphism
M(x) = αx. Let τ = M or τ = Fr. Recall the multiplication rule of O from
equation (1). Note that ϕ(τx, τy) = ϕ(x, y). It follows that (ab)^τ = a^τ b^τ for
a, b ∈ O where τ acts on O by e^x → (e^x)^τ = e^{τx}. Since the derivations D(a ∧ b)
is defined in terms of multiplication in O, it follows that (D(a ∧ b)c)^τ = D(a^τ ∧ b^τ)c^τ
for all a, b, c ∈ O and thus, by linearity,

\[(D(w)e)^τ = D(w^τ)e^τ\] for all \(w ∈ ∧^2 \text{Im}(O), c ∈ O\).

The two dimensional subspaces of F^2 (as a F_2-vector space) correspond to lines of
P^2(F_2). So we shall call these subspaces lines. There are three lines containing 1.
Let \(\{0, 1, x, y\} ⊆ F^2\) be one of these. Define

\[Δ = e^x ∧ e^y + (e^x ∧ e^y)' + (e^x ∧ e^y)'' ∈ ∧^2 \text{Im}(O)\].

Note that the element ±Δ is independent of choice of the line and choice of the
ordered pair \((x, y)\), since the Frobenius action permutes the three lines containing
1, and interchanging \((x, y)\) changes Δ by a sign. To be specific, we choose \((x, y) =
(α, α^3)\). Then

\[Δ = e_1 ∧ e_3 + e_2 ∧ e_6 + e_4 ∧ e_5\].

6. Lemma. (a) \(\ker(D)\) has a basis given by \(Δ, Δ^M, \cdots, Δ^M^6\).

(b) One has \([e_13, e_26] = 0\).

Proof. (a) Let \(w ∈ ∧^2 \text{Im}(O)\) and \(c ∈ O\). Since \(D(w)c = 0\) implies \(D(w^M)c^M = 0\),
it suffices to show that \(D(Δ) = 0\). Lemma 3 implies that if \(\{0, 1, x, y\} ⊆ F^2\) is a
subset corresponding to a line in \(P^2(F_2)\), then \(D(e^x ∧ e^y)e_1^1 = 0\), since \(x + y = 1\).
So \(D(Δ)e_1^1 = 0\). Since \(Δ' = Δ\), the equation \(D(Δ)e_1^1 = 0\) implies \(0 = D(Δ')(e^x)' =
D(Δ)e^x\). So it suffices to show that \(D(Δ)\) kills \(e_1^1\) and \(e_3^3\). This is an easy
calculation using Lemma 3. This proves that \(Δ, Δ^M, \cdots, Δ^M^6 ∈ \ker(D)\). One
verifies that these seven elements are linearly independent.

(b) Write \(X = [e_13, e_26]\). From part (a), we know that \(e_13 + e_26 + e_{45} = 0\). It
follows that \([e_13, e_{26}] = [e_{26}, e_{45}]\), that is, \(X\) is Frobenius invariant. So it suffices
to show that \(X\) kills \(e_0, e_1, e_3\). The equation \(Xe_0 = 0\) is immediate. Verifying
\(Xe_1 = 0\) is an easy calculation using Lemma 3. The calculation for \(e_3\) is identical
to the calculation for \(e_1\) since \(e_1 = -e_3\) and \(e_26 = -e_{62}\). \(\square\)
7. Remark. We want to sketch a conceptual argument to prove \([e_{13},e_{26}] = 0\) that was pointed out to us by the referee. For \(v \in \mathbb{O}\), let \(L_v\) (resp. \(R_v\)) denote the left (resp. right) multiplication by \(v\).

Let \(x, y\) be distinct element of \(\mathbb{F}_8^*\). Let \(\mathcal{H} = \langle e^x, e^y \rangle = R e^0 + R e^x + R e^y + R e^{x+y}\) be the quaternion subalgebra of \(\mathbb{O}\) spanned by \(e^x\) and \(e^y\). Let \(\mathcal{H}^\perp\) be the orthogonal complement of \(\mathcal{H}\) with respect to the standard positive definite form on \(\mathbb{O}\). First observe that lemma 3 can be rephrased as saying that \(D(e^x \wedge e^y)\) acts as \(\text{ad}_{e^x e^y}\) on \(\mathcal{H}\) and as \(L_{e^x e^y}\) on \(\mathcal{H}^\perp\).

Write \(d_1 = e_{13}\) and \(d_2 = e_{26}\). Let \(\mathcal{H}_1 = \langle e_1, e_3 \rangle\) and \(\mathcal{H}_2 = \langle e_2, e_6 \rangle\). So \(\mathcal{H}_1\) and \(\mathcal{H}_2\) are two quaternion subalgebras of \(\mathbb{O}\) corresponding to the two lines \(l_1 = \{0, 1, \alpha, \alpha^3\}\) and \(l_2 = \{0, 1, \alpha^2, \alpha^6\}\). Since \(e_1 e_3 = e_2 e_6 = e_0\), we have

\[d_j |_{\mathcal{H}_j} = \text{ad}_{e_0}, \quad d_j |_{\mathcal{H}_j^\perp} = L_{e_0},\]

for \(j = 1, 2\). Pick a standard basis element \(e_r\) of \(\mathbb{O}\). Then \(e_r \in \mathcal{H}_1\) or \(\mathcal{H}_1^\perp\). Also \(e_r \in \mathcal{H}_2\) or \(\mathcal{H}_2^\perp\). For purpose of illustration, suppose \(e_r \in \mathcal{H}_1 \cap \mathcal{H}_1^\perp\). Then

\[(d_1 d_2 - d_2 d_1) e_r = d_1 L_{e_0} e_r - d_2 e_0 \text{ad}_{e_0} e_r = \text{ad}_{e_0} L_{e_0} e_r - L_{e_0} \text{ad}_{e_0} e_r = 0\]

where the second equality holds because \(L_{e_0}, R_{e_0}\) (and consequently \(\text{ad}_{e_0}\)) preserves \(\mathcal{H}_1\) and \(\mathcal{H}_2\) and hence preserves \(\mathcal{H}_1^\perp\) and \(\mathcal{H}_2^\perp\). If \(e_r \in \mathcal{H}_1 \cap \mathcal{H}_2\) or \(e_r \in \mathcal{H}_1^\perp \cap \mathcal{H}_2\) or \(e_r \in \mathcal{H}_1^\perp \cap \mathcal{H}_2^\perp\), the argument is identical. This proves \([d_1, d_2] = 0\). Of course this argument actually shows that if \(z \in \mathbb{F}_8^*\) and if \(\{0, z, x, y\}\) and \(\{0, z, u, v\}\) are any two lines containing \(z\), then \([D(e^x \wedge e^y), D(e^u \wedge e^v)] = 0\).

8. Definition (The roots and coroots). Lemma 6 implies that \((C e_{13} + C e_{26})\) is an abelian subalgebra of \(\mathfrak{g}\). In section 10, we shall obtain a decomposition of \(\mathfrak{g}\) as direct sum of one dimensional simultaneous eigenspaces for \(\text{ad}_{e_{13}}\) and \(\text{ad}_{e_{26}}\) and it would follow that \((C e_{13} + C e_{26})\) is a maximal abelian subalgebra or a Cartan subalgebra. We fix this Cartan subalgebra and call it \(H\). Fix a pair of coroots

\[H_{\pm \beta} = \pm H_\beta = \mp ie_{13}\]

in \(H\). Using Frobenius action, we obtain the six coroots \(\pm \{H_\beta, H'_\beta, H^2_\beta\}\) corresponding to the short roots. The six coroots corresponding to the long roots are \(\pm \{H_\gamma, H'_\gamma, H^2_\gamma\}\) where

\[H_{\pm \gamma} = \pm H_\gamma = \pm \frac{1}{3}(e_{13} - e_{26})\].

We shall define a basis of \(\mathfrak{g}\) containing \(H_\beta\) and \(H_\gamma\). The scaling factors like \(\frac{1}{3}\) are chosen to make sure that the structure constants of \(\mathfrak{g}\) with respect to this basis are integers and are smallest possible. Define roots \(\beta, \gamma\) such that

\[
\begin{pmatrix}
\beta(H_\beta) & \beta(H_\gamma) \\
\gamma(H_\beta) & \gamma(H_\gamma)
\end{pmatrix} = \begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}
\]

is the Cartan matrix of \(\mathfrak{g}_2\). So \(\{\beta, \gamma\}\) is a pair of simple roots with \(\beta\) being the short root; see Figure 1. Let \(\Phi_{\text{short}}\) be the set of six short roots and let \(\Phi\) be the set of twelve roots of \(\mathfrak{g}\). Note that the Frobenius acts on \(H\) by anti-clockwise rotation of 120–degrees and complex conjugation acts by negation.

Once the roots and coroots have been fixed, the weight space decompositions of the two smallest irreducible representations of \(\mathfrak{g}\) can be found by simultaneously diagonalizing the actions of \(H_\beta\) and \(H_\gamma\). These weight spaces are described below.
9. **The standard representation:** Write $V = \text{Im}(O) \otimes \mathbb{C}$. This is the standard representation of $\mathfrak{g}$. Define the vectors $v_0, v_{\pm \beta}, v_{\pm \beta}', v_{\pm \beta}''$ in $V$ by choosing

$$v_0 = e_0 \text{ and } v_{\pm \beta} = (\pm ie_1 + e_3).$$

One easily verifies that $v_0$ spans the weight space $V_\psi$ for each short root $\psi$. See Figure 2. One has the weight space decomposition: $V = \mathbb{C}v_0 \oplus \bigoplus_{\psi \in \Phi_{\text{short}}} \mathbb{C}v_\psi$.

10. **The adjoint representation:** If $\psi$ is a short root of $\mathfrak{g}$, define

$$E_\psi = \frac{1}{2} D(v_0 \wedge v_\psi).$$

If $\nu$ is a long root of $\mathfrak{g}$, then there exists a unique short root $\psi$ such that $\nu = \psi - \psi'$. Define

$$E_\nu = \frac{1}{6} D(v_\psi \wedge v_\psi').$$

One easily verifies that $E_\rho$ spans the root space $\mathfrak{g}_\rho$ for each root $\rho \in \Phi$. One has the root space decomposition: $\mathfrak{g} = H \oplus \bigoplus_{\rho \in \Phi} \mathbb{C}E_\rho$. Note that

$$E_\beta = \frac{1}{8} D(v_0 \wedge v_\beta) = \frac{1}{4} D(e_0 \wedge (ie_1 + e_3)) = \frac{1}{2}(-ie_{10} + e_{03}),$$

and

$$E_\gamma = \frac{1}{6} D(v_{-\beta} \wedge v_\beta') = \frac{1}{6} D((-ie_1 + e_3) \wedge (ie_2 + e_6)) = \frac{1}{6}(e_{12} + e_{36} - i(e_{23} + e_{16})).$$

To write down the other $E_\rho$’s, apply the $\mathbb{Z}/6\mathbb{Z}$ symmetry generated by complex conjugation and Frobenius.

11. **Theorem.** The set $\{H_\beta, H_\gamma\} \cup \{E_\rho; \rho \in \Phi\}$ is a Chevalley basis of $\mathfrak{g}$.

   **Remark on proof.** Checking that these generators of $\mathfrak{g}$ obey the commutation rules dictated by the root space decomposition is a routine verification using their action on the standard representation $V$ as described in remark 13. Because of the visible $\mathbb{Z}/6\mathbb{Z}$ symmetry of our construction, only few cases need to be checked. □

12. **Remark.** Identify $(\wedge^2 \text{Im}(O) \otimes \mathbb{C})$ with $\mathfrak{so}_7(\mathbb{C})$ in the standard manner (see [FH], page 303) so that $e_i \wedge e_j$ gets identified with the skew symmetric matrix $2(E_{ij} - E_{ji})$ where $E_{ij}$ is the matrix with rows and columns indexed by $\mathbb{Z}/7\mathbb{Z}$ whose only nonzero entry is $1$ in the $(i, j)$-th slot. Let $\nu$ be a long root. Write $\nu = \psi - \psi'$ for a short root $\psi$. It is curious to note that

$$[D(v_0 \wedge v_\psi), D(v_0 \wedge v_{-\psi'})]_\mathfrak{g} = 4[E_\psi, E_{-\psi'}]_\mathfrak{g} = 12E_\nu = -D[v_0 \wedge v_\psi, v_0 \wedge v_{-\psi'}]_{\mathfrak{so}_7(\mathbb{C})},$$

even though $-D$ is not a Lie algebra homomorphism.
Figure 2. Basis for weight spaces in the standard representation \( \text{Im}(\mathfrak{g}) \otimes \mathbb{C} \). The numbers next to the dashed arrows indicate the scalars involved in action of some of the \( E_\psi \)'s as stated in equation (3). The rest can be worked out from weight consideration and \( \mathbb{Z}/6\mathbb{Z} \) symmetry.

13. Remark (Action of the Chevalley basis on the standard representation). The action of the vectors \( \{ E_\rho : \rho \in \Phi \} \) on the weight vectors \( \{ v_0 \} \cup \{ v_\psi : \psi \in \Phi_{\text{short}} \} \) is determined up to scalars by weight consideration since \( [\mathfrak{g}_\rho, V_\rho] \subseteq V_{\rho+\rho'} \) and each weight space \( V_\rho \) is at most one dimensional. The non-trivial scalars are determined by the following rules: Let \( \psi \) be a short root and let \( \rho \) be a root such that \( \psi + \rho \) is also a short root. Then

\[
E_\psi v_0 = v_\psi, \quad E_\psi v_{-\psi} = -2v_0, \quad \text{and} \quad E_\rho v_\psi = \epsilon v_{\rho+\psi}
\]

where \( \epsilon = 1 \) if \( v_{\rho+\psi} \) is equal to \( v_\rho' \) or \( -v_\rho'' \) and \( \epsilon = -1 \) otherwise. In other words, \( \epsilon = 1 \) if and only if the movement from \( \psi \) to \( (\rho + \psi) \) in the direction of \( \rho \) defines an anti-clockwise rotation of angle less than \( \pi \) around the origin. The relations in equation (3) are easily verified using Lemma 3. Only a few relations need to be checked, because of the \( \mathbb{Z}/6\mathbb{Z} \) symmetry. The nontrivial scalars involved in this action are indicated in Figure 2 next to the dashed arrows. For example, the \(-2\) next to the horizontal arrow means that \( E_\beta v_{-\beta} = -2v_0 \).

14. The irreducible representations of \( \mathfrak{g}_2 \): We finish by describing the finite dimensional irreducible representations of \( \mathfrak{g} \) in terms of the standard representation \( V \). This was worked out in [HZ]. The description given below follows quickly from the results of [HZ].

Fix the simple roots \( \{ \beta, \gamma \} \) as in Figure 1. Then the fundamental weights of \( \mathfrak{g} \) are \( \mu_1 = -\beta'' \) and \( \mu_2 = -\gamma' \). For each non-negative integer \( a, b \), let \( \Gamma_{a,b} \) denote the finite dimensional irreducible representation of \( \mathfrak{g} \) with highest weight \( (a\mu_1 + b\mu_2) \). The two smallest ones are the standard representation \( V = \Gamma_{1,0} \) and the adjoint representation \( \mathfrak{g} = \Gamma_{0,1} \).

Let \( \lambda \) be the Young tableau having two rows, corresponding to the partition \((a+b,b)\). Then \( \Gamma_{a,b} \) can be realized as a subspace of the Weyl module \( S_\lambda(V) \); see [HZ], Theorem 5.5. From [F], chapter 8, recall that the vectors in \( S_\lambda(V) \) can be represented in the form

\[
w = \frac{w_{1,1}}{w_{2,1}} \frac{w_{1,2}}{w_{2,2}} \cdots \frac{w_{1,b}}{w_{2,b}}
\]

where \( w_{i,j} \in V \), modulo the following relations:
Interchanging the two entries of a column negates the vector $w$.

Interchanging two columns of the same length does not change $w$.

For each $1 \leq j \leq b$ and $j < k \leq a + b$, let $z_1$ (resp. $z_2$) be the vector obtained from $w$ by interchanging $w_{1,k}$ with $w_{1,j}$ (resp. $w_{2,j}$). Then $w = z_1 + z_2$.

These relations are the exchange conditions of [F], page 81, worked out in our situation.

The natural surjection from $\otimes^{a+b} V \to S_\lambda(V)$ induces the $g$–action on $S_\lambda(V)$. Note that the highest weight of $\Gamma_{a,b}$ is $(a\mu_1 + b\mu_2) = (a + b)(-\beta'') + b\beta'$. From Figure 2, recall that $v''_{\beta} = -ie_4 + e_5$ and $v'_{\beta} = ie_2 + e_6$. Let $w_\lambda \in S_\lambda(V)$ be the vector written in the form given in equation (4) whose first row entries are all equal to $(-ie_4 + e_5)$ and whose second row entries are all equal to $(ie_2 + e_6)$. Then we find that $w_\lambda$ has weight $(a\mu_1 + b\mu_2)$. So $\Gamma_{a,b} = U(g)w_\lambda$, and $w_\lambda$ is the highest weight vector of $\Gamma_{a,b}$.

Acknowledgement: I would like to thank Jonathan Smith and Jonas Hartwig for many interesting discussions and helpful suggestions. I am grateful to the referee for suggesting many improvements and correcting several mistakes. In particular, the conceptual argument for Lemma 6(b) sketched in remark 7 is due to him.

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