Math 520: Methods of Applied Mathematics II

Homework Problem Set 5

Due April 30

1. If (a, b) is any bounded interval in $\mathbb{R}$, prove that $W^{1,p}(a, b)$ is compactly embedded in $L^p(a, b)$ for any $1 < p \le \infty$. (Suggestion: Use the Arzela-Ascoli theorem.)

2. If $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth enough boundary, find a solution of the wave equation problem
$$u_{tt} - \Delta u = 0 \qquad x \in \Omega \quad t > 0$$
$$u(x, t) = 0 \qquad x \in \partial\Omega \quad t > 0$$
$$u(x, 0) = f(x) \quad u_t(x, 0) = g(x) \qquad x \in \Omega$$
in the form
$$u(x, t) = \sum_{n=1}^{\infty} c_n(t)\psi_n(x)$$
where $\{\psi_n\}_{n=1}^{\infty}$ are the Dirichlet eigenfunctions of $-\Delta$ in Ω.

3. Let A, B be symmetric linear operators on a Hilbert space $\mathbf{H}$ with $D(A) \cap D(B)$ dense in $\mathbf{H}$, and define the Rayleigh quotient
$$J(x) = \frac{(Ax, x)}{(Bx, x)} \qquad x \in D(J) = \{x \in D(A) \cap D(B) : x \neq 0\}$$
Show that if $x \neq 0$ is a critical point of J, that is
$$\frac{d}{d\epsilon} J(x + \epsilon y)|_{\epsilon=0} = 0 \qquad \forall y \in D(J)$$
then $Ax = \lambda Bx$ for some λ. (For simplicity you can assume that $(Ax, y), (Bx, y)$ are real for any x, y.)

4. The area of a surface obtained by revolving the graph of $y = u(x), 0 < x < 1$ about the x axis, is
$$J(u) = 2\pi \int_0^1 u(x)\sqrt{1 + u'(x)^2}\, dx$$
Assume that u is required to satisfy $u(0) = a, u(1) = b$ where $0 < a < b$.
   (a) Find the Euler-Lagrange equation for the problem of minimizing this surface area.
   (b) Show that
$$\frac{u(u')^2}{\sqrt{1 + (u')^2}} - u\sqrt{1 + (u')^2}$$
   is a constant function for any such minimal surface (see Proposition 13.37 in text).

5. We say that u is a weak solution of the Neumann problem

$$-\Delta u = f \quad x \in \Omega \qquad \frac{\partial u}{\partial n} = 0 \quad x \in \partial\Omega \qquad \text{(NP)}$$

(as usual, Ω is a bounded domain in $\mathbb{R}^n$) if $u \in H^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H^1(\Omega)$$

Prove an existence theorem if $f \in L^2(\Omega)$, $\int_\Omega f(x)\, dx = 0$ in the following way:

(a) Appropriately modify the above definition and show that for any $\epsilon > 0$ there exists a unique weak solution u_ϵ of

$$-\Delta u + \epsilon u = f \quad x \in \Omega \qquad \frac{\partial u}{\partial n} = 0 \quad x \in \partial\Omega$$

(b) Show that $\int_\Omega u_\epsilon(x)\, dx = 0$ for any such ϵ.

(c) Show that there exist $u \in H^1(\Omega)$ such that $u_\epsilon \to u$ weakly in $H^1(\Omega)$ as $\epsilon \to 0$, and u is a solution of (NP).

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Homework Problem Set 4

Due March 12

1. Show that if $S \in \mathcal{B}(\mathbf{H})$ and T is compact, then TS and ST are also compact. In algebraic terms this means that the set of compact operators is an *ideal* in $\mathcal{B}(\mathbf{H})$.

2. If $T \in \mathcal{B}(\mathbf{H})$ and T^*T is compact, show that T must be compact. Use this to show that if T is compact then T^* must also be compact.

3. Show that if $\mathbf{H}$ is not separable then $0 \in \sigma_p(T)$ for any compact operator T.

4. Show that the operator
$$Tu(x) = \frac{1}{x}\int_0^x u(y)\,dy$$
is not compact on $L^2(0,1)$. (Suggestion: show that $u(x) = x^\alpha$ is an eigenfunction for every $\alpha > 0$.)

5. Show that the operator $T \in \mathcal{B}(\ell^2)$ given by $Tx = \left(0, \frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$ is compact. Prove that T has no eigenvalues.

6. The concept of a Hilbert-Schmidt operator can be defined abstractly as follows. If $\mathbf{H}$ is a separable Hilbert space, we say that $T \in \mathcal{B}(\mathbf{H})$ is Hilbert-Schmidt if
$$\sum_{n=1}^{\infty} ||Tu_n||^2 < \infty \tag{1}$$
for some orthonormal basis $\{u_n\}_{n=1}^{\infty}$ of $\mathbf{H}$.

   (a) Show that if T is Hilbert-Schmidt then the sum (1) must be finite for *any* orthonormal basis of $\mathbf{H}$. Suggestion: If $\{v_n\}_{n=1}^{\infty}$ is another orthonormal basis, then
$$\sum_{n=1}^{\infty} ||Tv_n||^2 = \sum_{n,m=1}^{\infty} |(Tv_n, u_m)|^2 = \sum_{n,m=1}^{\infty} |(v_n, T^*u_m)|^2 = \sum_{n,m=1}^{\infty} |(u_n, T^*u_m)|^2$$

   (b) Show that a Hilbert-Schmidt operator is compact.

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Homework Problem Set 3

Due February 19

1. Let $T \in \mathcal{B}(\mathbf{H})$ and let $R_\lambda(T)$ be the resolvent operator of T. Show that $\|R_\lambda(T)\| \to 0$ as $\lambda \to \infty$.
2. Let T be a bounded linear operator on a Hilbert space $\mathbf{H}$. If $q(x)$ is a polynomial, show that
$$\sigma(q(T)) = \big\{q(\lambda) \mid \lambda \in \sigma(T)\big\}$$
3. Let $T \in \mathcal{B}(\mathbf{H})$ be a self-adjoint operator. If $\sigma(T)$ contains exactly one point λ, show that $T = \lambda I$.
4. Give an example of a bounded operator $T \in \mathcal{B}(\ell^2)$ such that $T \neq 0$ but $\sigma(T) = \{0\}$.
5. Let $Tu(x) = u(x-1)$ be the translation operator on $L^2(\mathbb{R})$. Show that
$$\sigma(T) = \sigma_c(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$
(Suggestion: to study the equation $Tu - \lambda u = f$ take the Fourier transform of both sides.)
6. If $\lambda \neq \pm 1, \pm i$ show that λ is in the resolvent set of the Fourier transform. (Suggestion: First show that if $Tu - \lambda u = f$ then $T^4 u = \lambda^4 u + \lambda^3 f + \lambda^2 T f + \lambda T^2 f + T^3 f$, and then use the fact that $T^4 = I$ if T is the Fourier transform.)
7. Let $T : \ell^\infty \to \ell^\infty$ be defined by $x \mapsto (\xi_2, \xi_3, \ldots)$, where x is given by $x = (\xi_1, \xi_2, \ldots)$.
   (a) If $|\lambda| \leq 1$, show that $\lambda \in \sigma_p(T)$ and find the corresponding eigenspace.
   (b) If $|\lambda| > 1$, show that $\lambda \in \rho(T)$.

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Homework Problem Set 2
Due February 8

1. Show that an isometric linear operator $T : \mathbf{H} \to \mathbf{H}$ which is not unitary maps the Hilbert space $\mathbf{H}$ onto a proper closed subspace of $\mathbf{H}$.

2. Let T, S be densely defined linear operators on a Hilbert space. If $T \subset S$, show that $S^* \subset T^*$.

3. Let $\mathbf{H}$ be a complex Hilbert space and $T : \mathcal{D}(T) \subset \mathbf{H} \to \mathbf{H}$ linear and densely defined in $\mathbf{H}$. Show that T is symmetric if and only if $\langle Tx, x\rangle$ is real for all $x \in \mathcal{D}(T)$.

4. If a linear operator T is densely defined in $\mathbf{H}$ and its adjoint is defined on all of $\mathbf{H}$, show that T is bounded.

5. (Problem 9.3 in text) The operator $R_\lambda = (\lambda I - T)^{-1}$ is called the *resolvent operator* of T.

   (a) Prove the resolvent identity

   $$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \quad \lambda, \mu \in \rho(T)$$

   (b) Deduce from this that R_λ, R_μ commute.

   (c) Show also that T, R_λ commute for $\lambda \in \rho(T)$.

6. (Problem 10.15 in text) Let $\varphi \in \mathbf{H} = L^2(\mathbb{R})$ be any nonzero function and define the linear operator

   $$Tu = \left(\int_{-\infty}^{\infty} u(x)\, dx \right) \varphi$$

   on the domain $D(T) = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

   (a) Show that T is unbounded and densely defined.

   (b) Show that T^* is not densely defined, more specifically show that T^* is the zero operator with domain $\{\varphi\}^\perp$. (It then follows from a theorem stated in class that T is not closeable.)

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Homework Problem Set 1

Due February 1

1. If $T_1, T_2 \in \mathcal{B}(\mathbf{X})$ show that $||T_1 + T_2|| \leq ||T_1|| + ||T_2||$ and that $||T_1 T_2|| \leq ||T_1|| \, ||T_2||$.

2. (Problem 8.15 in text) If $\mathbf{H}$ is a Hilbert space and $S, T \in \mathcal{B}(\mathbf{H})$, show that

   (i) $(S+T)^* = S^* + T^*$

   (ii) $(ST)^* = T^* S^*$

3. Assume that $T \in \mathcal{B}(\mathbf{X}, \mathbf{Y})$, $\mathbf{X}$ is a Banach space and there exists $m > 0$ such that $\|Tx\| \geq m\|x\|$ for any $x \in \mathbf{X}$. Show that the range of T is a closed subspace of $\mathbf{Y}$.

4. An operator $T \in \mathcal{B}(\mathbf{H})$ is said to be *normal* if it commutes with its adjoint, i.e. $T^*T = TT^*$. Show that $T \in \mathcal{B}(\mathbf{H})$ is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in \mathcal{B}(\mathbf{H})$.

5. If $\{T_n\}_{n \in \mathbb{N}} \subseteq \mathcal{B}(\mathbf{H})$ is a sequence of normal linear operators and $T_n \to T$, show that T is a normal linear operator.

6. Which of the following operators $K : L^2\big([a,b]\big) \to L^2\big([a,b]\big)$ have finite rank and which do not?

   (i) $Kf(x) = \sum_{j=1}^{n} \phi_j(x) \int_a^b \psi_j(s) f(s) ds$, where $\phi_j(x)$ and $\psi_j(x)$ are functions in $L^2\big([a,b]\big)$.

   (ii) $K\phi(x) = \int_a^x \phi(s) ds$.

7. Let $T : L^2(-\infty, \infty) \to L^2(-\infty, \infty)$ with $Tf(x) = f(x+1)$. Find the adjoint operator of T.