American Mathematical Society
Fall 2005 Central Section Meeting
University of Nebraska–Lincoln, October 21-23, 2005

Special Session: Association schemes and related topics

1. **Session Program**
   *AS=association scheme, CoCo=coherent configuration, DRG=distance-regular graph, SRG=strongly regular graph.

Session I: Saturday, Oct. 22, 08:00 - 11:30 AM

08:00AM **P. Terwilliger**, DRGs and the quantum affine algebra $U_q(\hat{sl}_2)$
08:30AM **M. MacLean**, Subconstituent algebra of a bipartite DRG
09:00AM **S. Miklavic**, On bipartite Q-polynomial DRGs
09:30AM **M. Lang**, Pseudo Idempotents and Bipartite Q-Polynomial DRGs
10:00AM **J. D. H. Smith**, Permutation representations, characters and set partitions
10:30AM **A. Chan**, Type-II matrices for the Hamming schemes
11:00AM **C. Godsil**, Type-II matrices.

Session II: Saturday, Oct. 22, 3:00 - 6:40 PM

3:00PM **P.-H. Zieschang**, On ASs of odd order
3:30PM **J. Kim**, Characterization of some SRGs on 64 vertices
4:00PM **M. Klin**, CoCo of type $AK(2)$ on 16 points and its merging
4:30PM **A. Woldar**, AS on 28 points as merging of a half-homogeneous CoCo.
5:10PM – 6:00PM (AMS invited talk)
6:10PM **Aleksandar Jurišić***, Triangle-free DRGs with an eigenvalue multiplicity equal to the valency and diameter 3
6:30PM **S. Y. Song**, Terwilliger algebras of certain wreath product ASs.
6:50PM (Going out for dinner together.)

Session III: Sunday, Oct. 23, 08:00 - 11:30 AM

08:00AM **Etsuko Bannai**, On Euclidean designs
08:30AM **K. Johnson**, Gelfand-Tzetlin bases and ASs
09:00AM **R. Hein**, Homogeneous P-polynomial integral table algebras
09:30AM **G. Bhattacharyya**, Polygonal designs: existence and construction.
10:00AM **J. Ma**, A new method of finding fusion schemes
10:30AM **B. Curtin**, Duality of Bose-Mesner algebras
11:00AM **W. Martin**, Some remarks on imprimitive cometric (Q-polynomial) ASs.

2. **Abstracts**

- **Etsuko Bannai*** (Kyushu Univ., Japan) (Sun. 8am)
  On Euclidean designs.

  Abstract

  The concept of Euclidean designs was defined by Neumaier and Seidel in 1988 as a generalization of spherical designs. Delsarte and Seidel proved the Fisher type lower bounds for the cardinality of a Euclidean $2e$-design and that of an antipodal Euclidean $(2e+1)$-design, and then they gave definitions of Euclidean tight designs. In this talk we examine the definitions of tightness more carefully, and present some new results on the basic properties of tight designs. Moreover, we give the complete classification of the following designs: (1) Euclidean tight 2-designs, (2) Euclidean tight 4-designs supported by 2 concentric spheres, (3) antipodal Euclidean tight 3-designs, and (4) antipodal Euclidean tight 5-designs supported by 2 concentric spheres. Finally we discuss the non-rigidity of Euclidean designs and give some results which have been obtained jointly with Eiichi Bannai and Djoko Suprijanto.

- **Gargi Bhattacharyya*** (Iowa State Univ.) (Sun. 9:30 AM)
  Polygonal Designs: Existence and Construction

  Abstract

  Polygonal designs form a special class of partially balanced incomplete block designs. We resolve the existence problem for polygonal designs with various parameter sets and present several construction methods with blocks of small size (jointly with J. Hegeman, J. Kim, J. Langford, and S. Y. Song).

- **Ada Chan** (York Univ. CA) (Sat. 10:30 AM)
  Type-II matrices for the Hamming Schemes

  Abstract

  An $n \times n$ matrix W is type II if

  $$(W^{-1})_{ij} = \frac{1}{nW_{ji}}.$$

  Using Nomura's construction, each type-II matrix W yields a Bose-Mesner algebra, also called the Nomura algebra of W. The computation of the Nomura algebra of a type-II matrix is straightforward, if your computer is powerful enough.

  More interesting questions are: is a given Bose-Mesner algebra the Nomura algebra of a type-II matrix? If yes, what can we say about the type-II matrix? We will investigate these questions for the Hamming schemes in the talk.

- **Brian Curtin** (Univ. of South Florida) (Sun. 10:30AM)
  Duality of Bose-Mesner algebras.

Abstract

We discuss duality of Bose-Mesner algebras. Recently we have considered hyper-duality as an extension of formal duality to the subconstiuent (Terwilliger) algebra of a Bose-Mesner algebra. We shall survey some of our results concerning formal and hyper-duality and discuss some open problems.

- **Chris Godsil*** (Univ. of Waterloo, CA) (Sat. 11:00AM)
  Type-II Matrices.

Abstract

The *Schur product* $M \circ N$ of two $m \times n$ matrices M and N is the $m \times n$ matrix with ij-entry $M_{i,j}N_{i,j}$. If the entries of M are non-zero, the *Schur inverse* $M^{(-)}$ satisfies $M \circ M^{(-)} = J$, where J is the all-ones matrix. Finally, an $n \times n$ matrix W is a type-II matrix if

$$W^{(-)} = n(W^{-1})^T.$$

(Hadamard matrices provide one class of examples.)

Type-II matrices have interesting connections to link invariants and to association schemes. In this talk I will summarize some of their basic properties, and show how they arise in connection with a range of combinatorial objects (jointly with Ada Chan).

- **Robert Hein*** (Northern Illinois Univ.) (Sun. 9:00AM)
  Homogeneous P-polynomial Integral Table Algebras.

Abstract

The standard, monotonic algebras of the title are classified, as are the subset that have integer dual degrees. (Here, "standard" means that all column sums of the generating tridiagonal matrix are equal, and "monotonic" means that the superdiagonal entries of the matrix are nondecreasing while the subdiagonal entries are nonincreasing.) Most of the resulting algebras do not occur as adjacency algebras of distance-regular graphs (jointly with Harvey Blau).

- **Kenneth Johnson** (Penn. State - Ogontz) (Sun. 8:30AM)
  Gelfand-Tzetlin bases and association schemes.

Abstract

Recently the representation theory of the symmetric group has been explained by Vershik and Okounkov in terms of Young-Jucys-Murphy elements and Gelfand Tsetlin bases. These elements of the group algebra of S_n are defined in terms of sums of involutions as follows:

$$X_0 = 0,\ X_1 = (12), ...,\ X_i = (1i) + (2i) + ... + ((i-1)i), ...$$

Although the elements $\{X_i\}$ are not in the center of $\mathbb{C}S_n$ they form a commutative algebra. Now the representation theory of any group is usually presented in terms of the class algebra, which as is well-known forms a commutative coherent configuration. I will explore the relationship between the two approaches and generalizations to arbitrary association schemes.

- **Aleksandar Jurišić***, (IMFM, Slovenia) (Sat. 6:10PM)
  Triangle-free distance-regular graphs with an eigenvalue multiplicity equal to the valency and diameter 3

  Let Γ be a triangle-free distance-regular graph with diameter $d = 3$ and assume that Γ has an eigenvalue t with multiplicity k. Let u and v be the remaining nontrivial eigenvalues of Γ. We first parametrize the entries of the intersection array with eigenvalues $\{k, t, u, v\}$. If Γ is bipartite and antipodal, then $\Gamma = K_{k+1,k+1}$ with a perfect matching deleted. If Γ is nonbipartite, then $t = -1$ if and only if Γ is antipodal, in which case $k = -uv$, $c_2 = -(u + v)$ and the size of antipodal classes is $r = -(u - 1)(v - 1)/(u + v)$. (The case of Γ being bipartite and not antipodal is impossible.)

  Suppose now Γ is primitive, hence $t \neq -1$. Let w_0, w_1, w_2, w_3 be the cosine sequence corresponding to t. We show Γ is formally self-dual (hence Q-polynomial and 1-homogeneous). Furthermore, the smallest positive eigenvalue and the largest negative eigenvalue resp. are kw_2 and kw_3 resp. and are not equal to t. If these eigenvalues are u and v resp., we also have $(k - v)(k + v) = (k + vt)(v - t)$, all the eigenvalues are integral and we have either $k > t > u > 0 > v$ or $k > u > 0 > v > t$.

  Let $x, y \in V\Gamma$ be two adjacent vertices and $z \in \Gamma_2(x) \cap \Gamma_2(y)$. Then the intersection number $\tau_2 := |\Gamma(z) \cap \Gamma_3(x) \cap \Gamma_3(y)|$ is independent of the choice of vertices x, y and z. We classify all the graphs with $b_2 = \tau_2$ and obtain that the only example of such graphs is the coset graph of the doubly truncated binary Golay code. In particular, we rule out an infinite family of otherwise feasible intersection arrays. (Joint work with Jack Koolen, Arjana Žitnik.)

- **Joohyung Kim*** (Iowa State Univ.) (Sat. 3:30PM)
  Characterization of some strongly regular graphs on 64 vertices.

Abstract

We will discuss some characterization problems of strongly regular graphs with parameters (64, 28, 12, 12) coming from Hamming cubes, mutually orthogonal Latin squares, orthogonal arrays, and Hadamard matrices (jointly with S. Y. Song).

- **Mikhail Klin*** (Ben-Gurion Univ. of Negev, Israel)(Sat. 4:00PM)
  Coherent configuration of type $AK(2)$ on 16 points and its mergings.

Abstract

Recently Muzychuk, following ideas of Wallis and Fon-Der-Flaass, introduced a new wide class of coherent configurations, which provide a large amount of mergings, leading to association schemes. We consider a particular case of such WFDF configurations, which are denoted by $AK(n)$. Any configuration of type $AK(n)$ contains $n+2$ fibers, each of size n^2.

We investigate the unique (up to isomorphism) configuration of type $AK(2)$ and its corresponding coherent algebra W. We construct groups of usual, color, and algebraic automorphisms of W and classify with respect to these groups all homogeneous and all algebraic coherent subalgebras of W. A special attention is payed to twins, that is to pairs of algebraically isomorphic, but not combinatorially isomorphic association schemes.

In such manner we provide an unified explanation for a number of known and new phenomena related to coherent configurations of order 16 (jointly with Mikhail Muzychuk and Sven Reichard).

- **Michael Lang** (Bradley Univ.) (Sat. 9:30AM)
  Pseudo Idempotents and Bipartite Q-Polynomial Distance-Regular Graphs.

Abstract

Pseudo idempotents are a generalization of primitive idempotents, associated with any complex number rather than only eigenvalues. They allow characterization of some interesting combinatorial and algebraic conditions. I will describe a few of the results that are known about pseudo idempotents, including their behavior in bipartite Q-polynomial distance-regular graphs.

- **Jianmin Ma** (Emory Univ.) (Sun. 10:00AM)
  A new method of finding fusion schemes.

Abstract

We present a method to construct commutative fusion schemes for a non-commutative scheme X. It exploits "double product" homomorphic images of the underlying adjacency algebra of X. This leads to "lifting" issues that are addressed combinatorially.

This method is applied to the centralizer algebra of $S_b^l \leqslant S_{lb}$, where S_n denotes the symmetric group of degree n and S_b^l is the direct product of l copies of S_b. The case $l = b = 3$ is studied in detail and several new commutative fusion schemes are found in this 55-dimensional algebra.

- **Mark S. MacLean*** (Seattle Univ.) (Sat. 8:30AM)
  The subconstituent algebra of a bipartite distance-regular graph; thin modules with endpoint two.

Abstract

We consider a bipartite distance-regular graph Γ with diameter $D \geq 4$, valency $k \geq 3$, intersection numbers b_i, c_i and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Let A_i denote the i^{th} distance matrix of Γ. Fixing a vertex x, let E_i^* denote the projection onto the i^{th} subconstituent of Γ, and let T denote the Terwilliger algebra of Γ with respect to x. Let W denote a thin irreducible T-module with endpoint 2. Observe E_2^*W is a 1-dimensional eigenspace for $E_2^*A_2E_2^*$; let η denote the corresponding eigenvalue. Let $d = \lfloor D/2 \rfloor$. It is known $\tilde{\theta}_1 \leq \eta \leq \tilde{\theta}_d$ where $\tilde{\theta}_1 = -1 - b_2b_3(\theta_1^2 - b_2)^{-1}$, $\tilde{\theta}_d = -1 - b_2b_3(\theta_d^2 - b_2)^{-1}$. For $\tilde{\theta}_1 < \eta < \tilde{\theta}_d$ we obtain the following results. We show the dimension of W is $D - 1$. We find two bases for W. We show each basis is orthogonal (with respect to the Hermitean dot product) and we compute the square norm of each basis vector. We find the matrix representing the adjacency matrix with respect to each basis. We find the transition matrix relating our two bases for W (jointly with P. Terwilliger).

- **William Martin*** (Worchester Polytech Institute) (Sun. 11:00AM)
  Some remarks on imprimitive cometric (Q-polynomial) association schemes.

Abstract

Suzuki (J. Algebraic Combin., 1998) gave a classification of imprimitive cometric association schemes. He proved that such a scheme is either Q-bipartite, Q-antipodal or satisfies one of two special conditions. In this talk, we first investigate structural conditions on such association schemes. If $\sum_{i \in S} A_i = I_w \otimes J_r$ where $1 < r, w < |X|$, then we prove that $r = 2$ when the scheme is Q-bipartite,

$w \leq m_1 + 1$ when the scheme is Q-antipodal, and more. We investigate vanishing intersection numbers in each case. We obtain most results by looking at the eigenspace geometry of the scheme. We also give parameter sets for potential cometric schemes that are not distance-regular graphs and we include a construction of some new imprimitive cometric schemes (jointly with Jason S. Williford).

- **Stefko Miklavic** (Univ. Primorska, Slovenia) (Sat. 9:00AM)
  On bipartite Q-polynomial distance-regular graphs.

Abstract

Let G denote a bipartite Q-polynomial distance-regular graph with vertex set X and diameter $D \geq 3$. Let V denote the vector space over real numbers consisting of column vectors with real entries and rows indexed by X. For $z \in X$, let $\hat{z}$ denote the vector in V with a 1 in the z-coordinate, and 0 in all other coordinates. Fix $x, y \in X$ such that $d(x, y) = 2$. For $0 \leq i, j \leq D$ we define $w_{ij} = \sum \hat{z}$, where the sum is over all $z \in X$ such that $d(x, z) = i$ and $d(y, z) = j$. We define $W = \text{span}\{w_{ij} \mid 0 \leq i, j \leq D\}$. In this talk we consider the space $MW = \text{span}\{mw \mid m \in M, w \in W\}$, where M is the Bose-Mesner algebra of G. We obtain our results about MW using Terwilliger's "balanced set" characterization of the Q-polynomial property.

Finally, let $\theta_0, \theta_1, \cdots, \theta_D$ denote the Q-polynomial ordering of the eigenvalues of G. It is well known that this sequence satisfies

$$\theta_{i-1} + \theta_{i+1} = \beta\theta_i \qquad (1 \leq i \leq d-1)$$

for some real scalar β. Let q denote a complex scalar such that $q + q^{-1} = \beta$. Using the idea of Terwilliger we give $q + q^{-1}$ as a simple rational expression involving the intersection numbers and some other combinatorial coefficients.

- **Jonathan D. H. Smith** (Iowa State Univ.) (Sat. 10:00AM)
  Permutation representations, characters and set partitions.

Abstract

The concept of a group permutation representation is extended to quasigroups and left quasigroups. Permutation matrices are replaced by more general stochastic matrices. Many aspects of the classical theory of group permutation representations, including Burnside's "Orbit-Counting" Lemma and the Burnside algebra, extend naturally to the new context. In the quasigroup case, a permutation character is associated with each permutation representation. Representations of the most elementary left quasigroups, namely sets with right projection, lead to questions about set partitions.

- **Sung Y. Song** (Iowa State Univ.)(Sat. 6:30PM)
  Terwilliger algebras of certain wreath product schemes.

Abstract

We will discuss the structure of the Terwilliger algebras of association schemes obtained from the wreath product of complete graphs.

- **Paul Terwilliger*** (Univ. of Wisconsin) (Sat. 8:00AM)
  Distance-regular graphs and the quantum affine algebra $U_q(\widehat{sl}_2)$.

Abstract

Let Γ denote a distance-regular graph that is formally self dual and has classical parameters. For example, Γ is the distance-regular graph associated with the bilinear forms, the alternating forms, the Hermitean forms, or the quadratic forms. We display a natural action of the quantum affine algebra $U_q(\widehat{sl}_2)$ on the standard module of Γ. Our result shows that for Γ the subconstituent algebra is a homomorphic image of $U_q(\widehat{sl}_2)$. This is joint work with Tatsuro Ito.

- **Andrew Woldar*** (Villanova Univ.) (Sat. 4:30PM)
  Association Schemes on 28 Points as Mergings of a Half-Homogeneous Coherent Configuration.

Abstract

We enumerate, up to isomorphism, all association schemes on 28 points which arise as homogeneous fusions of the half-homogeneous coherent configuration $AP(2)$, thus providing a unified explanation of such schemes. Among those we encounter are the ones of pseudocyclic and quasithin type, plus two of pseudotriangular type. Configuration $AP(2)$ has a rich supply of algebraic automorphisms, which allows us to identify many small classes of fusions of $AP(2)$ which are algebraically isomorphic (inside $AP(2)$) but not combinatorially isomorphic. Given any such class of size at least 2, we call any pair of its members *twins*. Notable examples of twins are the triangular graph $T(8)$ paired with one of the Chang graphs, the (Schurian) pseudocyclic scheme of Mathon paired with the (non-Schurian) pseudocyclic scheme of Hollmann, and a Schurian quasithin scheme with 15 classes paired with a non-Schurian such scheme (jointly with M. Klin, M. Muzychuk, C. Pech, and P.-H. Zieschang).

- **Paul-Hermann Zieschang** (Univ. of Texas - Brownsville) (Sat. 3:00PM)
  On association schemes of odd order

  Abstract

  Let S be a finite association scheme and assume that, for each element s in S, $|s|$ has odd order. A theorem of Walter Feit and John Thompson says that S is solvable if S is thin. Using this theorem we shall prove that schurian simple schemes of odd order are primitive. We also give details about association schemes of order $p^\alpha q^\beta$.

3. **Meeting web-pages**:
   http://www.math.unl.edu/pi/events/ams2005
   http://www.ams.org/amsmtgs/
   http://www.math.unl.edu/pi/events/ams2005/travel.