Polygonal Designs: Existence and Construction

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Abstract

Polygonal designs form a special class of partially balanced incomplete block designs. We resolve the existence problem for polygonal designs with various parameter sets and present several construction methods with blocks of small sizes.

1 Introduction and preliminaries

Some results and notation that are used throughout the paper are collected in this section for convenience.

For positive integers v, b, k, and r with $2 < k < v$, $1 \leq r < b$, a (doubly regular) *incomplete block design* with parameters (v, b, k, r), is a pair $(V, \mathcal{B})$, where V is a set of v elements, called *points* or *varieties* and $\mathcal{B}$ is a collection of b k-element subsets of V called *blocks*, satisfying the condition: *each point appears in exactly* r *blocks.* An incomplete block design is said to be *balanced* if any two points are contained in exactly λ blocks together. It is called *partially balanced* if every pair of points occurs in a certain number of blocks depending on an association relation between the points.

A special class of partially balanced incomplete block designs called *polygonal designs* can be defined on a regular polygon. The set V forms a v-gon with vertex (point) set

$$V = \mathbb{Z}_v = \{0, 1, 2, \ldots, v-1\}.$$

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We define the distance $\delta(x,y)$ between points x and y to be the length of the shortest path connecting x and y on V. That is, for $x,y \in V$

$$\delta(x,y) := \min\{|x-y|,\ v-|x-y|\}, \quad \text{and thus } 0 \le \delta(x,y) \le \left\lfloor \frac{v}{2} \right\rfloor.$$

Definition 1.1 Let $V = \{0,1,\ldots,v-1\}$ be the point set of a regular v-gon, and let $m < (v-1)/2$. An incomplete block design $(V,\mathcal{B})$ with parameters (v,b,k,r) is called a *polygonal design with minimum interval* m, if any two points of V that are at distance $m+1$ or greater appear together in λ blocks while other pairs do not occur in the blocks at all. This design is denoted by $\mathrm{PD}(v,k,\lambda;m)$. We note that a $\mathrm{PD}(v,k,\lambda;0)$ is a balanced incomplete block design which is also known as a 2-(v,k,λ) design.

A number of authors have provided solutions to the existence and construction problems of polygonal designs for various combinations of v,b,k and λ. Some of the relevant references, almost all of which deal primarily with the case $m=1$, are as follows. Hedayat, Rao, and Stufken ([3]) introduced polygonal designs for the first time in 1988 as balanced sampling plans excluding contiguous units pertaining to finite population sampling. They showed that $v \ge 3k$ is a necessary condition for the existence of $\mathrm{PD}(v,k,\lambda;1)$. They also provided an iterative construction method by showing that if a $\mathrm{PD}(v,k,\lambda;1)$ exists, then a $\mathrm{PD}(v+3\alpha,k,\lambda';1)$ exists for any positive integer α. Stufken, Song, See and Driessel ([7]) showed that if a $\mathrm{PD}(v,k,\lambda;m)$ exists, then $b \ge v$ and $v \ge k(2m+1)$. They also showed that a $\mathrm{PD}(3k,k,\lambda;1)$ does not exist for any λ if $k \ge 5$. In regard to the construction of designs, Colbourn and Ling ([1], [2]) constructed all $\mathrm{PD}(v,k,\lambda;1)$ for $k=3$ and $k=4$. Stufken and Wright ([8]) constructed all possible $\mathrm{PD}(v,k,\lambda;1)$ with $k=5,6$ and 7, except possibly one, and several designs with block size 9 and 10.

In this paper, we study the polygonal designs with $v=k(2m+1)$ for an arbitrary m. We resolve the existence of polygonal designs with $v=k(2m+1)$ and $k=3$ completely. We present this result in Section 2. In Section 3, we show that if a $\mathrm{PD}(k(2m+1),k,\lambda;m)$ exists, then so does $\mathrm{PD}((k-1)(2m+1),k-1,\lambda';m)$ for some λ'. We also show that given a $\mathrm{PD}(v,k,\lambda;m)$ we can construct $\mathrm{PD}(v+(2m+1)\alpha,k,\lambda';m)$ for any positive integer α with some λ'. These results are a generalization of some of the results provided in [3] and [7]. In Section 4, we introduce a new construction method for $\mathrm{PD}(v,k,\lambda;m)$ by using a 'perfect (k,m)-grouping'. We also show that the inequality $k(k-1) \le 4(2m+1)$ holds in a $\mathrm{PD}(k(2m+1),k,\lambda;m)$. This result confirms the non-existence of $\mathrm{PD}(3k,k,\lambda;1)$ for $k \ge 5$ which was proved in [7].

Throughout the paper, we shall use $V = \{0,1,\ldots,v-1\}$ and $\mathcal{B} = \{B_1,B_2,\ldots,B_b\}$ for the point set and block set of a $\mathrm{PD}(v,k,\lambda;m)$ unless otherwise specified. Whenever we consider a block $B_i = \{b_{i1},b_{i2},\ldots,b_{ik}\} \in \mathcal{B}$, we shall assume that the points are ordered as $0 \le b_{i1} < b_{i2} < \cdots < b_{ik} \le v-1$. Given an integer a and a block B_i, by $B_i + a$ we shall denote the set $\{b_{i1}+a, b_{i2}+a, \ldots, b_{ik}+a\}$ where elements are reduced modulo v if needed. We shall also consider the differences $x-y$ (computed modulo v) as well as the distances

$\delta(x,y)$ between the points x and y. Let $B_i = \{b_{i1}, b_{i2}, ..., b_{ik}\}$, $i = 1, 2, \ldots, g$, be g blocks of size k based on $V = \{0, 1, \ldots, v-1\}$. Consider the $gk(k-1)$ differences

$$b_{ij} - b_{il}, \quad j, l \in \{1, 2, .., k\}, \ j \neq l, \quad 1 \leq i \leq g,$$

and call the multi-set

$$D = \{b_{ij} - b_{il} : \ j, l \in \{1, 2, \ldots, k\}, \ j \neq l, \ 1 \leq i \leq g\}$$

the *difference collection* of $G = \{B_1, B_2, \ldots, B_g\}$. Notice that we use the set notation, the curly bracket, for any collection of objects whether it is a set or a multi-set. We also often allow a design to have repeated blocks. We note that in a polygonal design PD$(v, k, \lambda; m)$, the difference collection of the entire block set $\mathcal{B}$ consists of each integer $m-1$ through $v-m-1$ equally $\frac{|\mathcal{B}|k(k-1)}{v-2m-1}$ times.

Definition 1.2 A collection $G = \{B_1, B_2, ..., B_g\}$ of k-subsets B_i of a v-set V is called a *generating set* of a PD$(v, k, \lambda; m)$ $(V, \mathcal{B})$ if

$$\mathcal{B} = \{B_i + a : \ 0 \leq a \leq t_i - 1, \ 1 \leq i \leq g\}$$

where t_i is the smallest positive integer such that $B_i + t_i \equiv B_i \pmod v$. A polygonal design is *cyclically generated* if and only if it has a generating set.

Note that for most blocks in the generating set t_i will simply be v. However, for blocks that are 'rotationally symmetric', t_i will be a proper divisor of v. When the differences are computed modulo v, each distance represented in a block, B_i, will occur a multiple of v/t_i times in that block. Suppose all t_i are equally v for a set G of blocks. Then it follows that G generates a polygonal design if and only if each of the differences $m+1, m+2, ..., v-(m+1)$ are represented in the difference collection of G exactly λ times. This statement can be modified slightly when G has a rotationally symmetric block, that is, there is a t_i that is a proper divisor of v.

We shall restrict our attention to cyclically generated designs. We note that there are no known polygonal designs that are not cyclically generated. It is obvious that whenever we have a polygonal design we can derive a cyclically generated polygonal design, perhaps, with a larger block set while keeping v, k and m constant.

For given m and k the polygonal designs PD$(v, k, \lambda; m)$ can possibly exist only when $v \geq (2m+1)k$ and $r = \lambda(v - 2m - 1)/(k-1)$. When we look at the class of polygonal designs with $v = (2m+1)k$ (in this case, we must have $b = (2m+1)^2\lambda$ and $r = (2m+1)\lambda$), the following lemma, which is Corollary 3.2 in [7] is very useful. (For the proof of this lemma we refer the reader to [7]).

Lemma 1.1 *Let $B = \{b_1, b_2, \ldots, b_k\}$ be a block in a PD$((2m+1)k, k, \lambda; m)$, where $b_1 < b_2 < \cdots < b_k$. If $1 \leq i < j \leq k$, then the difference $b_j - b_i$ lies in the set*

$$\{(j-i)(2m+1) - m, \ (j-i)(2m+1) - (m-1), \ \ldots, \ (j-i)(2m+1) + m\}.$$

2 Existence and construction of PD$(6m+3, 3, \lambda; m)$

In this section, we present a series of existence theorems, whose proofs provide construction methods for polygonal designs of block size $k = 3$ on $3(2m+1)$ points with the exception of the case when $m \equiv 2 \pmod 3$ and $\lambda = 1$. For this exceptional case a cyclically generated polygonal design does not exist.

For these constructions, we will be concerned with the distances (rather than differences), as by obtaining the distances of $m+1, m+2, ..., 3m+1$, we get all of the differences $m+1, m+2, ..., 5m+2$. For polygonal designs PD$(6m+3, 3, \lambda; m)$ we can write the possible sets of distances, contributed to the difference collection D of a generating set, by a single generating block in the form (d_1, d_2, d_3). For this to be a possible triple of distances we must have either $d_1 + d_2 + d_3 = 6m + 3$ or $d_1 + d_2 = d_3$. In the case that we are examining; i.e, with $k = 3$ and $v = 6m + 3$, $d_1 + d_2 = d_3$ is excluded as a possibility because $d_1 + d_2$ must be in $\{3m+2, 3m+3, \ldots, 5m+2\}$ due to Lemma 1.1. The following technical lemma will be used repeatedly.

Lemma 2.1 *The integers* $1, 2, 3, ..., 3n+1$ *excluding* $\lceil \frac{3n+1}{2} \rceil$ *can be partitioned into triples* (Y_1, Y_2, Y_3) *such that* $Y_1 + Y_2 = Y_3$.

Proof: In the following we present two side-by-side tables having headers Y_1, Y_2, and Y_3. Each row from each table presents a valid Y_1, Y_2, Y_3 triple. These tables give the required partitions where each integer appears in exactly one triple.

Case 1: n is even

Y_1	Y_2	Y_3
n	$n+1$	$2n+1$
$n-2$	$n+2$	$2n$
$n-4$	$n+3$	$2n-1$
$\vdots$	$\vdots$	$\vdots$
2	$\frac{3n}{2}$	$\frac{3n}{2}+2$

Y_1	Y_2	Y_3
$n-1$	$2n+2$	$3n+1$
$n-3$	$2n+3$	$3n$
$n-5$	$2n+4$	$3n-1$
$\vdots$	$\vdots$	$\vdots$
1	$\frac{5n}{2}+1$	$\frac{5n}{2}+2$

Case 2: n is odd

Y_1	Y_2	Y_3
n	$2n+1$	$3n+1$
$n-2$	$2n+2$	$3n$
$n-4$	$2n+3$	$3n-1$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
1	$\frac{5n+1}{2}$	$\frac{5n+1}{2}+1$

Y_1	Y_2	Y_3
$n-1$	$n+1$	$2n$
$n-3$	$n+2$	$2n-1$
$n-5$	$n+3$	$2n-2$
$\vdots$	$\vdots$	$\vdots$
2	$\frac{3n}{2}$	$\frac{3n}{2}+2$
$\vdots$	$\vdots$	$\vdots$

■

Theorem 2.2 PD$(6m+3,3,1;m)$ *exists for* $m \equiv 1 \pmod 3$.

Proof: To demonstrate this we partition the distances $m+1, m+2, ..., 3m+1$ into triples (d_1, d_2, d_3) such that $d_1+d_2+d_3 = 6m+3$ and each distance is in exactly one triple. First, rewrite the distances $m+1, m+2, ..., 3m+1$ as $(2m+1)-m$, $(2m+1)-(m-1)$, ..., $(2m+1)+m$.

We use Lemma 2.1 in order to partition the integers $1, 2, \ldots, m$ with $m = 3n+1$ into triples (Y_1, Y_2, Y_3) such that $Y_1+Y_2 = Y_3$. Let (Y_1, Y_2, Y_3) be one such triple in the partition of 1, 2, ..., m. We then see that the triples $((2m+1)-Y_1,\ (2m+1)-Y_2,\ (2m+1)+Y_3)$ and $((2m+1)+Y_1,\ (2m+1)+Y_2,\ (2m+1)-Y_3)$ are both valid distance triples, as both sum to $6m+3$. Therefore, for each triple (Y_1, Y_2, Y_3) of elements from $\{1, 2, ..., m\}$ we have two valid distance triples. Now we observe that every distance $m+1$ through $3m+1$ except for

$$2m+1-\left\lceil \frac{m}{2} \right\rceil, \quad 2m+1, \quad \text{and } 2m+1+\left\lceil \frac{m}{2} \right\rceil$$

appears in exactly one of the distance triples. However, the distances $2m+1-\lceil \frac{m}{2} \rceil$, $2m+1$ and $2m+1+\lceil \frac{m}{2} \rceil$ clearly form a final valid distance triple as they sum to $6m+3$. As all distances now appear in one distance triple we can form a generating set for the PD$(6m+3,3,1;m)$ by forming a generating block $\{1, d_1+1, d_1+d_2+1\}$ for each distance triple (d_1, d_2, d_3), showing that a PD$(6m+3,3,1;m)$ exists for all $m \equiv 1 \pmod 3$. ■

Theorem 2.3 PD$(6m+3,3,1;m)$ *exists for* $m \equiv 0 \pmod 3$.

Proof: In the following tables, triples $(X_1,\ X_2,\ X_3)$ are presented such that $X_1+X_2+X_3 = 0$. From each triple $(X_1,\ X_2,\ X_3)$ we simply form the distance triple $((2m+1)+X_1,\ (2m+1)+X_2,\ (2m+1)+X_3)$. Each of the necessary distances will appear once in the resulting triples and thus we can use them to form blocks which will generate PD$(6m+3,3,1;m)$.

Case 1: m is odd

X_1	X_2	X_3	X_1	X_2	X_3
$-\frac{m}{3}$	$-(\frac{m}{3}+1)$	$\frac{2m}{3}+1$	$-(\frac{m}{3}-1)$	$-(\frac{2m}{3}+1)$	m
$-(\frac{m}{3}-2)$	$-(\frac{m}{3}+2)$	$\frac{2m}{3}$	$-(\frac{m}{3}-3)$	$-(\frac{2m}{3}+2)$	$m-1$
$-(\frac{m}{3}-4)$	$-(\frac{m}{3}+3)$	$\frac{2m}{3}-1$	$-(\frac{m}{3}-5)$	$-(\frac{2m}{3}+3)$	$m-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-1	$-\frac{m+1}{2}$	$\frac{m+1}{2}+1$	-2	$-(\frac{5m+3}{6}-1)$	$\frac{5m+3}{6}+1$
$\frac{m}{3}$	$\frac{m+1}{2}$	$-\frac{5m+3}{6}$	$\frac{m}{3}-1$	$\frac{m}{3}+1$	$-\frac{2m}{3}$
$\frac{m}{3}-2$	$\frac{2m}{3}+2$	$-m$	$\frac{m}{3}-3$	$\frac{m}{3}+2$	$-(\frac{2m}{3}-1)$
$\frac{m}{3}-4$	$\frac{2m}{3}+3$	$-(m-1)$	$\frac{m}{3}-5$	$\frac{m}{3}+3$	$-(\frac{2m}{3}-2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$\frac{5m+3}{6}$	$-(\frac{5m+3}{6}+1)$	2	$\frac{m+1}{2}-1$	$-(\frac{m+1}{2}+1)$

Case 2: m is even

X_1	X_2	X_3	X_1	X_2	X_3
$-\frac{m}{3}$	$-(\frac{m}{3}+1)$	$\frac{2m}{3}+1$	$-(\frac{m}{3}-1)$	$-(\frac{2m}{3}+1)$	m
$-(\frac{m}{3}-2)$	$-(\frac{m}{3}+2)$	$\frac{2m}{3}$	$-(\frac{m}{3}-3)$	$-(\frac{2m}{3}+2)$	$m-1$
$-(\frac{m}{3}-4)$	$-(\frac{m}{3}+3)$	$\frac{2m}{3}-1$	$-(\frac{m}{3}-5)$	$-(\frac{2m}{3}+3)$	$m-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-2	$-\frac{m}{2}$	$\frac{m}{2}+2$	-1	$-\frac{5m}{6}$	$\frac{5m}{6}+1$
$\frac{m}{3}$	$\frac{m}{2}+1$	$-(\frac{5m}{6}+1)$	$\frac{m}{3}-1$	$\frac{m}{3}+1$	$-\frac{2m}{3}$
$\frac{m}{3}-2$	$\frac{2m}{3}+2$	$-m$	$\frac{m}{3}-3$	$\frac{m}{3}+2$	$-(\frac{2m}{3}-1)$
$\frac{m}{3}-4$	$\frac{2m}{3}+3$	$-(m-1)$	$\frac{m}{3}-5$	$\frac{m}{3}+3$	$-(\frac{2m}{3}-2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	$\frac{5m}{6}-1$	$-(\frac{5m}{6}+1)$	1	$\frac{m}{2}$	$-(\frac{m}{2}+1)$

This completes the proof. ■

Theorem 2.4 PD$(6m+3,3,2;m)$ *exists for* $m \equiv 2 \pmod 3$*.*

Proof: **Case 1:** m is even

Use Lemma 2.1 to partition the integers $1,2,...,(m-1)$ and then form two distance triples as before from each triple in this partition. By repeating this a second time we will have every distance appearing twice in the resulting triples with the exception of the distances

$$(2m+1)-m,\ (2m+1)-\frac{m}{2},\ (2m+1)+0,\ (2m+1)+\frac{m}{2},\ (2m+1)+m$$

which do not appear at all. Now form the distance triples

$$((2m+1)+\frac{m}{2},\ (2m+1)+\frac{m}{2},\ (2m+1)-m),$$
$$((2m+1)-\frac{m}{2},\ (2m+1)-\frac{m}{2},\ (2m+1)+m),\ \text{ and}$$
$$((2m+1)-m,\ (2m+1)+0,\ (2m+1)+m).$$

This leaves us with every distance appearing twice except for $2m+1$ which appears once. However, $3(2m+1)=6m+3$ and so we can form the rotationally symmetric block that only contributes a single distance of $2m+1$ to the difference collection. We now have each of the necessary distances represented twice and so we can form generating blocks for $\mathrm{PD}(6m+3,3,2;m)$ when $m\equiv 2(\text{mod } 3)$.

Case 2: m is odd

Observe that by increasing the second and third values of each triple in Lemma 2.1 by one we get a partition of the integers $1,2,....,m$ excluding $\frac{m+1}{3}$ and $\frac{m+1}{2}$ into triples (X_1,X_2,X_3) such that $X_1+X_2=X_3$. Using the previous partition of $1,2,...,m-1$ excluding $\frac{m-1}{2}$ and this new partition each one to form distance triples as before, we are left missing one instance of

$$(2m+1)-m,\ (2m+1)-\frac{m+1}{2},\ (2m+1)-\frac{m-1}{2},\ (2m+1)-\frac{m+1}{3},$$
$$(2m+1)+\frac{m+1}{3},\ (2m+1)+\frac{m-1}{2},\ (2m+1)+\frac{m+1}{2},\ (2m+1)+m$$

and two instances of $2m+1$. We now form the distance triples

$$((2m+1)-m,\ (2m+1)+\frac{m-1}{2},\ (2m+1)+\frac{m+1}{2}),$$
$$((2m+1)+m,\ (2m+1)-\frac{m-1}{2},\ (2m+1)-\frac{m+1}{2}),$$
$$((2m+1)-\frac{m+1}{3},\ (2m+1)+0,\ (2m+1)+\frac{m-1}{3})\text{ and}$$
$$((2m+1)+0,\ (2m+1)+0,\ (2m+1)+0).$$

Each distance is now represented exactly twice in this group of valid distance triples and so we can generate $\mathrm{PD}(6m+3,3,2;m)$ for all $m\equiv 2(\text{mod } 3)$. ■

Theorem 2.5 PD$(6m+3,3,3;m)$ *exists for all* m.

Proof: From the following table we can form the distance triples $((2m+1)+X_1,\ (2m+1)+X_2,\ (2m+1)+X_3)$. The resulting distance triples contain each of the distances $m+1,\ m+2,\ \ldots,\ 3m+1$ inclusive exactly 3 times and so can be used to construct a PD$(6m+3,3,3;m)$.

X_1	X_2	X_3
$-m$	0	m
$-(m-1)$	1	$m-2$
$-(m-2)$	2	$m-4$
$\vdots$	$\vdots$	$\vdots$
0	m	$-m$
1	$-m$	$m-1$
2	$-(m-1)$	$m-3$
$\vdots$	$\vdots$	$\vdots$
m	-1	$-(m-1)$

■

Remark 2.1 A PD$(v,k,\lambda;m)$ can be formed from PD$(v,k,1;m)$ by simply repeating each of the blocks in PD$(v,k,1;m)$ λ times. Similarly, PD$(v,k,\lambda;m)$ for $\lambda \geq 2$ can be formed from a linear combination of the blocks from PD$(v,k,2;m)$ and PD$(v,k,3;m)$. We thus have that PD$(6m+3,3,\lambda;m)$ exists for all combinations of λ and m except for when $m \equiv 2 \pmod 3$ and $\lambda = 1$. Here, a cyclically generated PD$(v,k,1;m)$ cannot exist for $m \equiv 2 \pmod 3$.

Next, we conclude the current section by introducing a way to construct polygonal designs of block size 3 using the λ-fold triple systems discussed in Chapter 2 of [4]. A λ*-fold triple system* is a pair $(V,\mathcal{B})$, where V is a finite set and $\mathcal{B}$ is a collection of 3-element subsets of V called triples such that each pair of distinct elements of V belongs to exactly λ triples of $\mathcal{B}$.

Theorem 2.6 *If there exists a cyclically generated* λ*-fold triple system with* N $(N \geq 3)$ *points, then* PD$(N(2m+1),\ 3,\ \lambda;\ m)$ *exists.*

Proof: First take the distance triples (n_1,n_2,n_3) from the generating blocks for the λ-fold triple system. Observe that either $n_1+n_2+n_3 = N$ or $n_1+n_2 = n_3$ must hold. From each triple where $n_1+n_2+n_3 = N$ form the $2m+1$ distance triples of the form

$$(n_1(2m+1)+X_1,\ n_2(2m+1)+X_2,\ n_3(2m+1)+X_3)$$

where the values of X_1, X_2, and X_3 are taken from the rows of the following table. Observe that as

$$(n_1(2m+1)+X_1)+(n_2(2m+1)+X_2)+(n_3(2m+1)+X_3)=N(2m+1)$$

these will be valid distance triples.

X_1	X_2	X_3
$-m$	0	m
$-(m-1)$	1	$m-2$
$-(m-2)$	2	$m-4$
$\vdots$	$\vdots$	$\vdots$
0	m	$-m$
1	$-m$	$m-1$
2	$-(m-1)$	$m-3$
$\vdots$	$\vdots$	$\vdots$
m	-1	$-(m-1)$

Similarly, from the triples where $n_1+n_2=n_3$ form the $2m+1$ distance triples of the form

$$(n_1(2m+1)+Y_1,\ n_2(2m+1)+Y_2,\ n_3(2m+1)+Y_3)$$

where the values of Y_1, Y_2, and Y_3 are taken from the rows of the following table. These will also be valid distance triples as

$$(n_1(2m+1)+Y_1)+(n_2(2m+1)+Y_2)=(n_3(2m+1)+Y_3).$$

Y_1	Y_2	Y_3
m	0	m
$m-1$	-1	$m-2$
$m-2$	-2	$m-4$
$\vdots$	$\vdots$	$\vdots$
0	$-m$	$-m$
-1	m	$m-1$
-2	$m-1$	$m-3$
$\vdots$	$\vdots$	$\vdots$
$-m$	1	$-(m-1)$

■

3 Recursive construction of new designs from old

In this section we present two ways to construct another polygonal designs from given polygonal designs. Both are iterative construction methods and they produce designs over different size of point sets.

Theorem 3.1 *If a cyclic* $\mathrm{PD}(k(2m+1), k, \lambda; m)$ *exists then* $\mathrm{PD}((k-1)(2m+1), k-1, \lambda'; m)$ *exists where* $\lambda' = (k-1)\lambda$.

Proof: Take a block from the generating set for $\mathrm{PD}(k(2m+1), k, \lambda; m)$ and form the k-tuple of first distances that appear in the block:

$$\{b_2 - b_1, b_3 - b_2, ..., v + b_1 - b_k\} = \{d_1, d_2, ..., d_k\}.$$

Replace all of the k possible pairs of adjacent distances, d_i and d_{i+1} (or d_k and d_1) by $d'_i = d_i + d_{i+1} - (2m+1)$ to form the k-tuple $\{d'_1, d'_2, \ldots, d'_k\}$. Now consider the $(k-1)$-tuples of ordered distances that can be obtained from the k-tuple $\{d'_1, d'_2, \ldots, d'_k\}$ by removing a single distance d'_i for each $i = 1, 2, \ldots, k$. (There are k such $(k-1)$-tuples.) From each $(k-1)$-tuple, say $\{r_1, r_2, \ldots, r_{k-1}\}$, form the block $\{0,\ r_1,\ r_1 + r_2,\ \ldots, r_1 + r_2 + \cdots + r_{k-1}\}$ of size $k-1$ on $(k-1)(2m+1)$ points. Repeat this process for each generating block. The resulting blocks will form a generating set for $\mathrm{PD}((k-1)(2m+1), k-1, \lambda'; m)$.

This can be verified by counting the differences in the resulting blocks. First, though, we use Lemma 1.1. From this lemma we know that when $v = k(2m+1)$ the first differences (differences between adjacent points in a block) will be elements of the set

$$\{(2m+1) - m,\ (2m+1) - (m-1),\ ...,\ (2m+1) + m\},$$

the second differences will be elements of the set

$$\{2(2m+1) - m,\ 2(2m+1) - (m-1),\ ...,\ 2(2m+1) + m\},$$

and more generally that nth differences will be elements of the set

$$\{n(2m+1) - m,\ n(2m+1) - (m-1),\ ...,\ n(2m+1) + m\}.$$

Observe that for each time a first difference appeared in a generating block for the original design, it will appear $k-2$ times in the generating blocks for the resulting design. For each time a second difference appeared in the original generating blocks, it will appear unchanged $k-3$ times in the new generating blocks and will be reduced by $2m+1$ to the corresponding first difference once. For each time a third difference appeared in the original design, it will appear unchanged $k-4$ times and will be reduced by $2m+1$ to the corresponding second difference twice. This pattern continues on up to the $(k-1)$th differences which will be reduced all $k-2$ times by $2m+1$ in the resulting blocks to the corresponding $(k-2)$th difference. The result is that for every time the differences $m+1,\ m+2,\ ...,\ (k-1)(2m+1)+m$ appeared in the generating blocks for the original design, the differences $m+1,\ m+2,\ ...,\ (k-2)(2m+1)+m$ will appear $k-1$ times in the resulting generating blocks. ■

Theorem 3.2 *If* $\mathrm{PD}(v,k,\lambda;m)$ *exists then* $\mathrm{PD}(v+(2m+1),k,\lambda';m)$ *exists where* $\lambda' = (v-2m-1)\lambda$.

Proof: The new design can be constructed by taking each of the blocks from $\mathrm{PD}(v,k,\lambda;m)$ with points labeled $0,1,2,...,v-1$ and replacing each instance of $v-1$ with $v+2m-2$, each instance of $v-2$ with $v+2m-5$, each instance of $v-3$ with $v+2m-8$,..., and each instance of $v-m$ with $v-(m-1)$. These new blocks form a generating set for $\mathrm{PD}(v+(2m+1),k,\lambda';m)$.

This can be seen by observing that each of the differences $m+1, m+2, ..., v-(m+1)$ appears $v\lambda$ times in the difference collection of the blocks of $\mathrm{PD}(v,k,\lambda;m)$. The replacement described above will increase the difference, d, by $1,2,...,2m$ each λ times. The distance will be increased by $2m+1$ exactly $\lambda(d-m)$ times and will be unchanged the remaining $\lambda(v-d-m)$ times. It is straightforward to check that each of the distances $m+1, m+2, ..., (v+2m+1)-(m+1)$ appears $\lambda(v-2m-1)$ times in these new blocks. These blocks thus can be used as a generating set to form $\mathrm{PD}(v+2m+1,k,\lambda';m)$. ■

Remark 3.1 It follows that if there is a $\mathrm{PD}(v,k,\lambda;m)$, then there is a polygonal design with $v+(2m+1)\alpha$ points in blocks of size k for any positive α. This iterative method has been provided by John Stufken [6] as a generalization of the result of Hedayat, Rao and Stufken [3] for the case of $m=1$.

4 Construction of $\mathrm{PD}((2m+1)k,k,\lambda;m)$ from perfect (k,m)-grouping

In this section we introduce another technique that may be useful in construction and in the study of polygonal designs.

Definition 4.1 A *perfect* (k,m)*-grouping* is a collection of k-element multi-sets having elements from $\{0,1,...,m\}$ such that each of $\{1,2,\ldots,m\}$ appears as the distance between two elements of a multi-set precisely N times and 0 appears as the distance between two elements of a multi-set no more than $N/2$ times. The k-element multi-sets that form a perfect (k,m)-grouping will be called *groups*.

Example 4.1 (1) The collection $\{\{0,1,3\}\}$ of the single group $\{0,1,3\}$ is a perfect $(3,3)$-grouping with $N=1$, while $\{\{0,1,4,6\}\}$ is a perfect $(4,6)$-grouping with $N=1$.

(2) $\{\{0,0,2\},\{0,1,2\},\{0,1,2\}\}$ is a perfect $(3,2)$-grouping with $N=3$.

(3) The collections $\{\{0,0,1,1\}\}$ and $\{\{0,0,1,2,2\}\}$ are perfect $(4,1)$- and $(5,2)$-groupings respectively with $N=4$.

(4) $\{\{0,2,4\},\{0,1,4\},\{0,1,4\},\{0,1,3\}\}$ is a perfect $(4,4)$-grouping with $N=3$.

Theorem 4.1 $\mathrm{PD}((2m+1)k,k,\lambda;m)$ *exists for some* λ *if and only if there exists a perfect* (k,m)*-grouping.*

Proof: Suppose that $\mathrm{PD}((2m+1)k,k,\lambda;m)$ exists for some λ. According to Lemma 1.1, given a point b_i in an arbitrary block, the next point b_{i+1} must be an element of the set $\{b_i+(m+1),\ b_i+(m+2)\ ,...,\ b_i+(3m+1)\}$. More generally, point b_j must belong to

$$\{b_i+(2m+1)(j-i)-m,\ b_i+(2m+1)(j-i)-(m-1),\ ...,\ b_i+(2m+1)(j-i)+m\}.$$

Now select the point b_r in the block such that $b_r-b_i-(2m+1)(r-i)$ is minimized and apply Lemma 1.1 again. We thus have that an arbitrary point b_j must belong to the set

$$\{b_r+(2m+1)(j-r)-m,\ b_r+(2m+1)(j-r)-(m-1),\ ...,\ b_r+(2m+1)(j-r)+m\}.$$

However, as the value of $b_r-b_i-(2m+1)(r-i)$ was minimized by our selection of b_r, b_j cannot fall in the first half of these values with respect to b_r and thus b_j must belong to the set

$$\{b_r+(2m+1)(j-r),\ b_r+(2m+1)(j-r)+1,\ ...,\ b_r+(2m+1)(j-r)+m\}.$$

This allows us to write each point, b_j, in the block in the form $b_r+(2m+1)(j-r)+a_j$ where a_j takes on one of the values $0,1,...,m$. By writing the points of each block in this form it follows that if a multi-set $\{a_1,a_2,...,a_k\}$ is formed for each block in the polygonal design then by collecting all such multi-sets we can form a perfect (k,m)-grouping.

Conversely, suppose we have a perfect (k,m)-grouping. Take a particular group $\{a_1,a_2,...,a_k\}$ and form the $(k-1)!$ blocks of the form

$$\{f(a_1),\ (2m+1)+f(a_2),\ ...,\ (k-1)(2m+1)+f(a_k)\}$$

where f is taken over all of the $(k-1)!$ permutations on the group $\{a_1, a_2, ..., a_k\}$ such that $f(a_1) = a_1$. When this is done for each group in the perfect (k,m)-grouping, all of the necessary differences $m+1, m+2, ..., v-(m+1)$ with **possible** exception of $(k-1)$ multiples of $2m+1$ will each be represented $N(k-2)!$ times in the difference collections from these $(k-1)!$ blocks. All of the differences that are multiples of $2m+1$ can be added to the difference collections a single time by the symmetric block

$$\{0,\ 2m+1,\ 2(2m+1),\ ...,\ (k-1)(2m+1)\}.$$

Thus, we can add some number of instances of this block to the previous blocks to make all necessary differences $m+1,\ m+2,\ ...,\ v-(m+1)$ be represented in the difference collections for these blocks precisely $N(k-2)!$ times. These blocks will thus form a generating set for PD$((2m+1)k, k, \lambda; m)$ where $\lambda = N(k-2)!$. ■

Theorem 4.2 $k(k-1) \leq 4(2m+1)$ *is a necessary condition for the existence of a perfect* (k,m)*-grouping.*

Proof: Suppose that there exists a perfect (k,m)-grouping where $k(k-1) > 4(2m+1)$. If this is the case, then the average number of times that each of the distances $1, 2, \ldots, m$ appears per group of the grouping must be greater than 4. We will develop a contradiction by demonstrating that if the distance m appears on average more than 4 times per group, then the distance 0 will appear more than half as many times as the distance m. This is not allowed by the definition of a perfect (k,m)-grouping.

We start by examining the number of times the distance m appears in a given group. Observe that m can only appear as the distance between the integers 0 and m. We let the number of instances of 0 in a particular group be denoted x and the number of instances of m in that group be denoted y. The number of times that m appears as a distance in this group will then be simply xy. Observe that the distance 0 will appear between two points in this group a minimum of $\frac{1}{2}(x(x-1) + y(y-1))$ times. We now find for what values of x and y the distance m will appear in the group at least twice as many times as the distance 0 by solving the inequality

$$xy \geq x(x-1) + y(y-1),$$

or equivalently

$$x + y - xy \geq (x-y)^2.$$

It is straightforward to show that the only positive integral solutions (x, y) to this inequality are $(2,2), (1,2), (2,1)$, and $(1,1)$. With the exception of $(x,y) = (1,1)$ all of these solutions yield strict equality. Thus, if the average number of times the distance m appears per group is greater than 4, there must be a group in which $xy > 4$ and thus in which the values of

x and y are not solutions to our inequality. This means that the distance m will appear in this group fewer than twice as many times as the distance 0. As the only type of group in which the distance m can possibly appear more than twice as many times as the distance 0 has $(x, y) = (1, 1)$, we must have $(x-y)^2 + xy - x - y$ blocks of this form to compensate for the group with $xy > 4$. The average number of times that the distance m appears per group in these groups is thus

$$\frac{xy + (x-y)^2 + xy - x - y}{1 + (x-y)^2 + xy - x - y}.$$

This must still be greater than 4 for it to be possible for the average number of times that m appears as a distance to be greater than 4. However it is straightforward to show that the expression is never greater than 4 when x and y are positive integers with $xy > 4$. This contradicts the original assumption thus proving that a perfect (k, m)-grouping does not exist for $k(k-1) > 4(2m+1)$. ■

Example 4.2 This bound of the above result is sharp for small k and m as perfect groupings exist for $(k, m) = (4, 1)$ and $(5, 2)$ both of which yield $k(k-1) = 4(2m+1)$. As we have seen in (3) of Example 4.1, each of these groupings consists of a single group of size k with $N = 4$, namely $\{\{0, 0, 1, 1\}\}$ and $\{\{0, 0, 1, 2, 2\}\}$. We note that the polygonal designs PD(12, 4, 4; 1) and PD(25, 5, 12; 2) generated by

$$G = \{\{0, 3, 7, 10\}, \{0, 4, 6, 10\}, \{0, 4, 7, 9\}\}$$

and

$$\begin{aligned} G = \{&\{0, 5, 11, 17, 22\}, \quad \{0, 5, 12, 16, 22\}, \quad \{0, 5, 12, 17, 21\}, \quad \{0, 6, 10, 17, 22\}, \\ &\{0, 6, 12, 15, 22\}, \quad \{0, 6, 12, 17, 20\}, \quad \{0, 7, 10, 16, 22\}, \quad \{0, 7, 10, 17, 21\}, \\ &\{0, 7, 11, 15, 22\}, \quad \{0, 7, 11, 17, 20\}, \quad \{0, 7, 12, 15, 21\}, \quad \{0, 7, 12, 16, 20\}\}, \end{aligned}$$

respectively are obtained from these perfect groupings. Notice that the examples provided here uses the method presented in the proof of Theorem 4.1 for converting between perfect groupings and polygonal designs, but we have removed the redundant generating blocks; so the number of blocks and λ for each designs have been reduced by a half.

Remark 4.3 While the bound of the previous theorem is sharp for small k and m as we have just seen in the above example, for larger values of k, this bound for m is not the strongest possible bound. We have a better bound that approaches the relationship $m \geq \frac{k^3}{48}$ for large k. The proof of this is given in the appendix.

Remark 4.4 As a consequence of the previous two theorems, PD($(2m+1)k, k, \lambda; m$) does not exists if $k(k-1) > 4(2m+1)$. That is, for $m = 1$, there exists no polygonal designs with

$v = 3k$ if $k \geq 5$ as it was proved in [7]. For $m = 2$, there exist no polygonal designs with $v = 5k$ if $k \geq 6$. Also it is an immediate consequence of the theorems that PD$(9, 3, 1; 1)$ and PD$(25, 5, 1; 2)$ are the only symmetric polygonal designs with $v = b = (2m + 1)k$.

Definition 4.2 A *natural perfect* (k, m)*-grouping* is any perfect (k, m)-grouping in which every group is a set, having no redundant elements; i.e., 0 does not appear as the distance between any two elements of any one group in the grouping.

For example, the perfect groupings listed in (1) and (4) of Example 4.1 are natural groupings. now the following theorem shows that if we have a perfect natural grouping, we can have another perfect grouping.

Theorem 4.3 *If there exists a natural perfect* (k, m)*-grouping then there exists a natural perfect* (k, m')*-grouping of size* m' *for all* $m' \geq m$.

Proof: We prove this by demonstrating that a natural perfect $(k, m + 1)$-grouping can be formed from a natural perfect (k, m)-grouping. This is done by taking every group in the natural perfect (k, m)-grouping and forming new groups by adding one to every element of the group that is greater than or equal to n. Do this for $n = 1, 2, ..., m + 1$ to form $m + 1$ new groups for each group in the natural perfect (k, m)-grouping. Each of the distances $1, 2, ..., m$ will be represented an equal number of times in the resulting groups and 0 will not appear at all. ■

For example, the perfect $(4, 4)$-grouping in (4) of Example 4.1 can be obtained from the perfect $(3, 3)$-grouping given in (1) of Example 4.1 in the manner described in the above proof.

Remark 4.5 For $k = 3$ or 4 there exist natural perfect (k, m)-groupings formed by a single k-element set. These are the perfect (3,3)-grouping and (4,6)-grouping illustrated in (1) of Example 4.1. Unfortunately, there are no natural perfect (k, m)-groupings consisting of a single (multi-)set for any other combinations of k and m. Natural perfect groupings with a single (multi-)set are very difficult to construct and as a result we have not found any natural perfect (k, m)-groupings for $k \geq 5$.

Remark 4.6 As we have found a natural perfect $(3, 3)$-grouping and $(4, 6)$-grouping, it thus follows from Theorems 4.1 and 4.2 that PD$((2m+1)k, k, \lambda; m)$ exists for some λ when $k = 3$ and $m \geq 3$ or $k = 4$ and $m \geq 6$. Additionally, for $k = 3$, $m = 1, 2$ and $k = 4$, $m = 1, 2, ..., 5$ we have manually confirmed the existence of such designs. Thus, PD$((2m + 1)3, 3, \lambda; m)$ and PD$((2m + 1)4, 4, \lambda; m)$ exist for some λ given any m.

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Appendix

Theorem For given k, let the minimum value of m for which a perfect (k,m)-grouping exists be denoted m_k. Then $\lim_{k\to\infty} \frac{m_k}{k^3} \geq \frac{1}{48}$.

Proof: We shall prove the statement only for the case when $m \equiv 3 (\mathrm{mod}\ 4)$, as m belongs to the other congruence classes modulo 4 can be treated similarly. Suppose there exists a perfect (k,m)-grouping that consists of R groups of the form $a_1, a_2, \ldots, a_k$ with $a_1 \leq a_2 \leq \cdots \leq a_k$.

Observe that the sum of the distances between points in a group $a_1, a_2, \ldots, a_k$ will be

$$(k-1)(a_1 + a_{k-1}) + (2k-4)(a_2 + a_{k-2}) + \cdots + (nk - n^2)(a_n + a_{k-n}) + \cdots$$
$$+ \left(\left(\frac{k-1}{2}\right) k - \left(\frac{k-1}{2}\right)^2 \right) \left(a_{\left(\frac{k-1}{2}\right)} + a_{\left(\frac{k+1}{2}\right)} \right)$$

where $n = 1, 2, \ldots, \frac{k-1}{2}$. We can rewrite this as

$$\left(\left(\frac{k-1}{2}\right) k - \left(\frac{k-1}{2}\right)^2 \right) \sum_{i=1}^{k-1} a_i - \sum_{n=1}^{\frac{k-1}{2}} \left(\left(\frac{k-1}{2}\right) k - \left(\frac{k-1}{2}\right)^2 - (nk - n^2) \right) (a_n + a_{k-n}).$$

Summing up this over all groups and replacing $\sum_{i=1}^{k-1} a_i$ by m (as $\sum_{i=1}^{k-1} a_i \leq m$), we obtain

$$\left(\left(\frac{k-1}{2}\right) k - \left(\frac{k-1}{2}\right)^2 \right) Rm - \sum_{\text{groups}} \sum_{n=1}^{\frac{k-1}{2}} \left(\left(\frac{k-1}{2}\right) k - \left(\frac{k-1}{2}\right)^2 - (nk - n^2) \right) (a_n + a_{k-n})$$

for an upper bound on the sum of all distances in all groups. Observe that when this expression achieves its maximum value it must be the case that $a_n, a_{k-n} \geq [\frac{n+1}{2}]$ for $n = 1, 2, \ldots, \frac{k-1}{2}$. We thus have another upper bound RU for the sum where

$$U := \left(\left(\frac{k-1}{2}\right) k - \left(\frac{k-1}{2}\right)^2 \right) m \; - \sum_{n=1}^{\frac{k-1}{2}} \left(\left(\frac{k-1}{2}\right) k - \left(\frac{k-1}{2}\right)^2 - (nk - n^2) \right) \cdot 2 \left[\frac{n}{2}\right].$$

Note that U is an upper bound for the average sum of the distances per group. This must be greater than or equal to the number of distances per group multiplied by the minimum average distance which is $\frac{k(k-1)}{2} \cdot \frac{m(m+1)}{2m+1}$. We thus have the inequality

$$U \geq \frac{k(k-1)}{2} \cdot \frac{m(m+1)}{2m+1}.$$

Multiplying both sides of this inequality by $2m + 1$, expanding, simplifying, dividing by k^3 and taking the limit as k approaches infinity yields

$$\lim_{k \to \infty} \frac{m_k}{k^3} \geq \frac{1}{48}.$$

■