Normally regular digraphs, association schemes and related combinatorial structures.

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Abstract

This paper reports the characteristics of and mutual relationships between various combinatorial structures that give rise to certain imprimitive nonsymmetric three-class association schemes. Nontrivial relation graphs of an imprimitive symmetric 2-class association scheme are $m \circ K_r$, (the union of $m$ copies of the complete graph on $r$ vertices) and its complement $m \circ \overline{K_r}$, (the complete $m$-partite strongly regular graph) for some positive integers $m$ and $r$. The set of nontrivial relation graphs of nonsymmetric three-class fission scheme of such a 2-class association scheme contains a complementary pair of oriented graphs of either $m \circ K_r$ or $m \circ \overline{K_r}$ depending on $m$ and $r$. For suitable parameters $m$ and $r$, these graphs arise from various combinatorial objects, such as, doubly regular tournaments, normally regular digraphs, skew Hadamard matrices, Cayley graphs of dicyclic groups and certain group rings. The construction and the characteristics of these objects are investigated combinatorially and algebraically, and their mutual relationships are discussed.

Keywords: Cayley graphs, regular team tournaments, skew symmetric Hadamard matrices, S-rings

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1 Introduction

A nonsymmetric three-class association scheme consists of three nontrivial associate relations, two of which are nonsymmetric relations. If $A$ is the adjacency matrix of one of the two nonsymmetric relations, then $A^t$ is the adjacency matrix of the directed graph associated with the other nonsymmetric relation. The undirected graph associated with the symmetric relation has adjacency matrix $J - I - A - A^t$. Being the relation graphs of an association scheme, it is required that these graphs are regular, in particular, $AJ = JA = kJ$, for some number $k$ and that

1. $AA^t = A^tA$ is a linear combination of $I$, $A$, $A^t$ and $J - I - A - A^t$, and

There are many combinatorial structures that possess either of these characteristics. This paper concerns the construction of such structures and studies their relationships. Among others, it discusses

- normally regular digraphs, i.e., regular directed graphs satisfying the condition 1, and
- doubly regular team tournaments, i.e., regular orientations of a complete multipartite graph satisfying the condition 2.

The paper also surveys the basic properties of other related objects including Cayley graphs and skew Hadamard matrices. It gives valuable descriptions of their mutual relationships and interpretations of certain characteristics of an object in terms of other structures. The organization of the paper is as follows.

In Section 2, we briefly recall basic terminologies and notations in graphs including matrix representations and automorphisms of graphs.

In Section 3, we describe our main objects, doubly regular tournaments, doubly regular orientations of complete multipartite strongly regular graphs, normally regular digraphs and association schemes. We recall known examples, their constructions and compare the characteristics of these objects. In particular, the doubly regular tournaments obtained from other doubly regular tournaments $T$ via the coclique extension (denoted $C_r(T)$) and a special way of ‘doubling’ and ‘augmenting’ by an additional pair of vertices (denoted $D(T)$), are equivalent to normally regular digraphs with certain parameter conditions. These objects are also related to certain class of doubly regular team tournaments and imprimitive symmetrizable 3-class association schemes.

In Section 4, we focus on the characterization of doubly regular $(m, r)$-team tournaments that may be viewed as the orientation of complete multipartite strongly regular graphs $m \circ K_r$. We then treat the $(m, 2)$-team tournaments as a special class in connection with symmetrizable 3-class association schemes.

In Section 5, the relationships between the first relation graphs of imprimitive nonsymmetric 3-class association schemes and special types of doubly regular tournaments $C_r(T)$ and $D(T)$ are discussed. In particular, it is shown that every imprimitive nonsymmetric 3-class association scheme for which the first relation graph is an orientation $m \circ K_r$ for $r = 2$, is characterized by the relation graph being isomorphic to either $C_2(T)$ or $D(T)$ for a suitable doubly regular tournament $T$. For $r \geq 3$, the first relation graphs of such association schemes are realized as special types of doubly regular graphs including $C_r(T)$ and $D(T)$.  

In Section 6, we study the groups acting transitively on a graph \( D(T) \) for some doubly regular tournament \( T \). In particular, we study the automorphism group of a vertex transitive doubly regular tournaments \( D(T) \). We observe that for any tournament \( T \) the graph \( D(T) \) has a unique involutory automorphism. We also show that the Sylow 2-subgroup of the automorphism group of certain \( D(T) \) is not cyclic but it is a generalized quaternion group.

In Section 7, Many normally regular digraphs arise from Cayley graphs, group rings and difference sets. We investigate the conditions on the connection set \( S \subset G \) under which the Cayley graph \( \text{Cay}(G, S) \) becomes a normally regular digraph. We then reformulate the condition in terms of group rings. We also show that if the Cayley graph of \( G \) with the generating set \( D \{1\} \) is isomorphic to \( D(T) \) for some doubly regular tournament \( T \), then \( D \) gives rise to a certain relative difference sets.

In Section 8, we recall the Noburo Ito’s conjecture [Ito97] in regard to Hadamard groups and matrices and consider S-rings over dicyclic groups to provide reinterpretation of the conjecture. Ito’s conjecture states that every dicyclic group of order \( 8t \), for some \( t \), is a Hadamard group. We consider possible values of \( t \) for which the dicyclic group of order \( 8t \) becomes a skew Hadamard group. Earlier in [Ito94], Ito proved that the Hadamard matrices corresponding to Paley tournaments \( P_{q} \) can be constructed from skew Hadamard groups. We state the result in terms of \( D(P_{q}) \) and prove it: that is, \( \text{SL}(2,q) \) acts as a group of automorphisms on \( D(P_{q}) \) and this group contains a dicyclic subgroup acting regularly on the vertex set. We then give examples illustrating that other groups beside the dicyclic groups may appear as regular subgroups of the automorphism groups of the graphs \( D(P_{q}) \) for some \( q \) through the computer search using GAP and GRAPE.

In Section 9, we give some concluding remarks.

2 Preliminaries

A directed graph \( \Gamma \) consists of a finite set \( V(\Gamma) \) of vertices and a set \( E(\Gamma) \subseteq \{(x, y) \mid x, y \in V(\Gamma)\} \) of arcs (or directed edges). If \((x, y) \in E(\Gamma)\), then we say that \( x \) and \( y \) are adjacent, \( x \) dominates \( y \), and that \( y \) is an out-neighbor of \( x \). (We also denote it as \( x \to y \).) The set of out-neighbors of \( x \) is the set \( N^{+}(x) = \{y \in V(\Gamma) \mid (x, y) \in E(\Gamma)\} \). The out-valency of \( x \) is the number \( |N^{+}(x)| \) of out-neighbors of \( x \). In-neighbors and in-valency are defined similarly. \( \Gamma \) is said to be regular of valency \( k \) if every vertex in \( \Gamma \) has in-valency and out-valency \( k \).

A graph can be viewed as a directed graph \( \Gamma \) for which \((x, y) \in E(\Gamma)\) if and only if \((y, x) \in E(\Gamma)\). An oriented graph is a directed \( \Gamma \) where at most one of the arcs \((x, y)\) and \((y, x)\) is in \( E(\Gamma) \). By a directed graph we usually refer to an oriented graph.

The adjacency matrix of a directed graph \( \Gamma \) with \( V(\Gamma) = \{x_1, \ldots, x_n\} \) is an \( n \times n \) matrix \( A \) with its \((i, j)\)-entry \( (A)_{i,j} = a_{ij} \) is defined by

\[
  a_{ij} = \begin{cases} 
    1 & \text{if } (x_i, x_j) \in E(\Gamma) \\
    0 & \text{otherwise}.
  \end{cases}
\]

The matrix \( J = J_n \) is the \( n \times n \) matrix with 1 in every entry. \( I = I_n \) denotes the \( n \times n \) identity matrix. Thus \( J - I \) is the adjacency matrix of the complete graph on \( n \) vertices.

In addition to the ordinary matrix multiplication we will use two other matrix products. Let \( A \) and \( B \) be \( n \times n \) matrices. Then the Schur-Hadamard product \( A \odot B \) is the \( n \times n \) matrix obtained
by the entry-wise multiplication: $(A \circ B)_{ij} = a_{ij}b_{ij}$. For an $n \times n$ matrix $A$ and an $m \times m$ matrix $B$ we define the Kronecker product of $A$ and $B$ to be the $nm \times nm$ block-matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$ 

The following lemma is immediate consequence of the definition.

**Lemma 2.1** Let $A$ and $B$ be $n \times n$ matrices and let $C$ and $D$ be $m \times m$ matrices. Then

$$(A \otimes C)(B \otimes D) = (AB) \otimes (CD).$$

A coclique extension of a given directed graph $\Gamma$ may be conveniently defined by Kronecker product of matrices as the following.

**Definition 2.2** Let $\Gamma$ be a directed graph with adjacency matrix $A$. The directed graph with adjacency matrix $A \otimes J_s$ is called a coclique extension of $\Gamma$, and denoted by $\mathcal{C}_s(\Gamma)$.

For a permutation $g$ of a set $X$, we use the notation $x^g$ for the image of $x \in X$ under $g$. An automorphism of a directed graph $\Gamma$ is a permutation of $V(\Gamma)$ so that $(x, y) \in E(\Gamma)$ if and only if $(x^g, y^g) \in E(\Gamma)$. The set of all automorphisms of $\Gamma$ forms a group called the automorphism group of $\Gamma$. Any subgroup of this group is called a group of automorphisms. A group $G$ of automorphisms is said to be semiregular if for any two vertices $x$ and $y$, there is at most one $g \in G$ so that $x^g = y$. $G$ is said to be transitive if for any two vertices $x$ and $y$, there is at least one $g \in G$ so that $x^g = y$. If $G$ is both semiregular and transitive then we say that $G$ is regular.

From a group $G$ and a set $S \subset G$, we define the Cayley graph of $G$ with connection set $S$ to be the (directed) graph $\text{Cay}(G, S)$ with vertex set $G$ and arc set $\{(x, y) \mid x^{-1}y \in S\}$. We always assume that $1 \not\in S$, where $1$ denotes the identity of the group, so that the Cayley graph has no loops. If $S = S^{(-1)}$, where $S^{(-1)}$ denotes the set $\{s^{-1} \mid s \in S\}$, then $\text{Cay}(G, S)$ is an undirected graph. If $S \cap S^{(-1)} = \emptyset$ then $\text{Cay}(G, S)$ is an oriented graph. It is known that a graph $\Gamma$ is a Cayley graph of a group $G$ if and only if $G$ is a regular group of automorphisms of $\Gamma$.

We will, in particular, consider Cayley graphs of dicyclic groups. The dicyclic group of order $4n$ is the group

$$\langle x, y \mid x^n = y^2, y^4 = 1, xyx = y \rangle.$$

The dicyclic group of order 8 is the quaternion group. A dicyclic group of order a power of 2 is also called a generalized quaternion group.

### 3 Main combinatorial structures

In this section, the structure and construction of various classes of doubly regular tournaments, doubly regular $(m, r)$-team tournaments and normally regular digraphs will be described. These are the combinatorial structures that are relevant to the three-class association schemes of our interest. Some of the mutual relationships between the above objects will be described as well.
3.1 Doubly regular tournaments

A tournament is a directed graph $T$ with the property that for any two distinct vertices $x$ and $y$, either $(x, y)$ or $(y, x)$, but not both, belongs to $E(T)$. In terms of the adjacency matrix $A$, a tournament is a directed graph with the property that $A + A^t + I = J$. If every vertex in a tournament $T$ with $n$ vertices has out-valency $k$ then every vertex has in-valency $n - k - 1$. Therefore, $k = n - k - 1$, i.e., $n = 2k + 1$; and so, such a tournament is a regular directed graph of valency $k$.

**Definition 3.1** A tournament $T$ is called doubly regular if it is regular and for every vertex $x$ in $T$ the out-neighbors of $x$ span a regular tournament.

**Lemma 3.2** Suppose that $T$ is a tournament with $|V(T)| = 4\lambda + 3$, and that its adjacency matrix $A$ satisfies

(a) one of $AJ = (2\lambda + 1)J$ and $JA = (2\lambda + 1)J$; and

(b) one of the equations $A^2 = \lambda A + (\lambda + 1)A^t$, $AA^t = \lambda J + (\lambda + 1)I$, and $A^tA = \lambda J + (\lambda + 1)I$.

Then all of these equations are satisfied and $T$ is a doubly regular tournament. Conversely, if $T$ is a doubly regular tournament then $|V(T)| = 4\lambda + 3$ and the adjacency matrix satisfies all of the equations in (a) and (b).

**Proof** If we multiply the equation $A + A^t = J - I$ by $J$ we get $JA + JA^t = (n - 1)J = (4\lambda + 2)J$. Thus $JA = (2\lambda + 1)J$ if and only if $JA^t = (2\lambda + 1)J$. The transpose of the second equation is $AJ = (2\lambda + 1)J$. Using the equations $AJ = JA = (2\lambda + 1)J$ and $A + A^t = J - I$, it can be shown that the equations involving $A^2$, $AA^t$ and $A^tA$ are equivalent.

Suppose that $T$ is a doubly regular tournament with $n$ vertices. Then it is regular of valency $k = \frac{n - 3}{2}$. The subgraph spanned by the out-neighbors of a vertex is regular of valency $\lambda = \frac{k + 1}{2} = \frac{n - 3}{4}$. Thus $n = 4\lambda + 3$. Since every vertex has out-valency $2\lambda + 1$, so $JA = (2\lambda + 1)J$. Let $x$ and $y$ be vertices with $(x, y) \in E(T)$. Then $y$ has out-valency $\lambda$ in $N^+(x)$; i.e., $x$ and $y$ have exactly $\lambda$ common out-neighbors. From this (and the fact that every vertex has out-valency $2\lambda + 1$) it follows that $AA^t = \lambda J + (\lambda + 1)I$.

Conversely, if $AJ = (2\lambda + 1)J$ then every vertex has out-valency $2\lambda + 1$. Thus $T$ is regular. If $AA^t = \lambda J + (\lambda + 1)I$ then for every vertex $x$ the out-neighbors of $x$ span a subtournament in which every vertex has out-valency $\lambda$. Thus this subtournament is regular and so $T$ is doubly regular. \qed

**Corollary 3.3** Let $x$ and $y$ be vertices in a doubly regular tournament $T$ with $4\lambda + 3$ vertices and suppose that $(x, y) \in E(T)$. Then the number of directed paths of length 2 from $x$ to $y$ is $\lambda$, the number of directed paths of length 2 from $y$ to $x$ is $\lambda + 1$, the number of common out-neighbors of $x$ and $y$ is $\lambda$ and the number of common in-neighbors of $x$ and $y$ is $\lambda$.

Paley [Pal33] constructed an infinite family of Hadamard matrices that corresponds to the following family of doubly regular tournaments. These tournaments are called Paley tournaments (cf. Theorems 3.4, 3.6, and 3.11 for the relationship to Hadamard matrices).
Theorem 3.4 ([Pal33]) Let \( q \equiv 3 \mod 4 \) be a prime power. Let \( \mathbb{F}_q \) denote the field of \( q \) elements and let \( Q \) be the set of nonzero squares in \( \mathbb{F}_q \). Then the graph \( P_q \) with \( V(P_q) = \mathbb{F}_q \) and \( E(P_q) = \{(x, y) \mid y - x \in Q\} \) is a doubly regular tournament.

Thus the Paley tournament \( P_q \) is the Cayley graph \( \text{Cay}(\mathbb{F}_q^+, Q) \) of the additive group \( \mathbb{F}_q^+ \).

Lemma 3.5 The Cayley graph \( \text{Cay}(\mathbb{F}_q^+, -Q) \) is isomorphic to \( P_q \).

Proof Multiplication by \(-1\) defines an isomorphism between \( \text{Cay}(\mathbb{F}_q^+, Q) \) and \( \text{Cay}(\mathbb{F}_q^+, -Q) \).

Theorem 3.6 Suppose that \( A \) is an adjacency matrix of a doubly regular tournament of order \( n \). Then the matrix

\[
B = \begin{bmatrix}
0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & \ddots & \ddots & \ddots & \ddots & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
A & \ddots & \ddots & \ddots & \ddots & \cdots & A + I \\
0 & \ddots & \ddots & \ddots & \ddots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & \cdots & A^t \\
0 & \ddots & \ddots & \ddots & \ddots & \cdots & 1
\end{bmatrix}
\]

is the adjacency matrix of a doubly regular tournament of order \( 2n + 1 \).

Proof It is clear that \( B \) is an adjacency matrix of a regular tournament of valency \( n \). An easy computation shows also that \( B \) satisfies the equation \( BB^t = (k + 1)I_{2n+1} + kJ_{2n+1} \) where \( k = (n - 1)/2 \).

Another family of doubly regular tournaments was found by Szekeres [Sze69].

Theorem 3.7 Let \( q \equiv 5 \mod 8 \) be a prime power. Let \( \mathbb{F}_q \) be the field of \( q \) elements and let \( Q \) be the unique subgroup of \( \mathbb{F}_q^* \) of index 4. Let \( Q, -Q, R, -R \) be the cosets of \( Q \). Then the graph \( S_q \) with vertex set

\[
\{v\} \cup \{x_i \mid x \in \mathbb{F}_q, i = 1, 2\}
\]

and arc set

\[
\{(v, x_1), (x_1, x_2), (x_2, v) \mid x \in \mathbb{F}_q\} \cup \{(x_1, y_1) \mid y - x \in Q \cup R\} \\
\cup \{(x_2, y_2) \mid y - x \in -Q \cup -R\} \cup \{(x_i, y_{3-i}) \mid y - x \in Q \cup -R, i = 1, 2\}
\]

is a doubly regular tournament of order \( 2q + 1 \).

Recently two new constructions of doubly regular tournaments as Cayley graphs were discovered by Ding and Yuan [DinY06] and by Ding, Wang and Xiang [DinWX07]. Doubly regular tournaments are also obtained from a class of Hadamard matrices.

Definition 3.8 An \( n \times n \) matrix \( H \) with entries \( \pm 1 \) is called a Hadamard matrix of order \( n \) if

\[
HH^t = nI.
\]
The condition $HH^t = nI$ implies that any two distinct rows of $H$ are orthogonal. Multiplication of any row or any column by $-1$ preserves this condition. Permutation of the rows and permutation of the columns also preserve the condition.

**Definition 3.9** Let $H_1$ and $H_2$ be Hadamard matrices of order $n$. Then we say that $H_1$ and $H_2$ are equivalent if $H_2$ can be obtained from $H_1$ by permuting rows, permuting columns and multiplying some rows and/or columns by $-1$.

It is known that if an Hadamard matrix of order $n > 2$ exists then $n \equiv 0 \text{ mod } 4$. The well known Hadamard matrix conjecture states that there exists an Hadamard matrix of order $n$ for every $n$ divisible by 4. We are interested in the following class of Hadamard matrices.

**Definition 3.10** An Hadamard matrix $H$ is called skew if

$$H + H^t = -2I.$$  

This implies that $H$ is skew if $H = (h_{ij})$ satisfies $h_{ij} = -h_{ji}$ for all $i \neq j$, and $h_{ii} = -1$ for all $i$.

A new skew Hadamard matrix can be obtained from an old by simultaneously multiplying the $i$th row and the $i$th column by $-1$ for any $i$. By repeating this procedure we can transform any skew Hadamard matrix to a skew Hadamard matrix of the following form.

$$H = \begin{bmatrix} -1 & 1 & \cdots & 1 \\ -1 \\ \vdots \\ -1 & B \end{bmatrix}$$  

(1)

where $B$ is $(n-1) \times (n-1)$ matrix with entries $\pm 1$. Reid and Brown [ReiB72] proved the following equivalence between skew Hadamard matrices and doubly regular tournaments.

**Theorem 3.11** Let $H$ and $B$ be matrices in equation (1). Then $H$ is a skew Hadamard matrix if and only if $A = \frac{1}{2}(B + J)$ is the adjacency matrix of a doubly regular tournament.

The proof follows by matrix computations and the properties of the adjacency matrix of a doubly regular tournament in Lemma 3.2.

If $A_1$ and $A_2$ are adjacency matrices of isomorphic doubly regular tournaments, then $A_2$ can be obtained from $A_1$ by applying the same permutation to rows and columns. Thus the Hadamard matrices constructed from $A_1$ and $A_2$ are equivalent. But it is also possible that non-isomorphic doubly regular tournaments correspond to equivalent skew Hadamard matrices.

### 3.2 Doubly regular orientations of $\overline{m \circ K_r}$

Let $m \circ K_r$ denote the disjoint union of $m$ copies of the complete graph on $r$ vertices, and let $\overline{m \circ K_r}$ denote its complement, i.e. the complete multipartite graph with $m$ independent sets of size $r$. Let $\Gamma$ be an orientation of $\overline{m \circ K_r}$, i.e., every edge $\{a, b\}$ in $\overline{m \circ K_r}$ is replaced by one of the arcs $(a, b)$ or $(b, a)$. Then we say that $\Gamma$ is an $(m, r)$-team tournament.
Definition 3.12 An \((m, r)\)-team tournament \(\Gamma\) with adjacency matrix \(A\) is said to be doubly regular if there exist integers \(k, \alpha, \beta, \gamma\) such that

i) \(\Gamma\) is regular with valency \(k\),

ii) \(A^2 = \alpha A + \beta A^t + \gamma (J - I - A - A^t)\).

Note that the equation \(A^2 = \alpha A + \beta A^t + \gamma (J - I - A - A^t)\) means that the number of directed paths of length 2 from a vertex \(x\) to a vertex \(y\) is \(\alpha\) if \((x, y) \in \Gamma\), \(\beta\) if \((y, x) \in \Gamma\) and \(\gamma\) if \(\{x, y\} \in m \times K_r\).

Proposition 3.13 Let \(T\) be a doubly regular tournament of order \(m = 4\lambda + 3\) and let \(r \in \mathbb{N}\). Then the coclique extension \(C_r(T)\) of \(T\) is a doubly regular \((m, r)\)-team tournament.

Proof Let \(A\) be the adjacency matrix of \(T\). Then, by Lemma 2.1, the adjacency matrix \(A \otimes J_r\) of \(C_r(T)\) satisfies

\[
(A \otimes J_r)^2 = A^2 \otimes J_r^2 = (\lambda A + (\lambda + 1)A^t) \otimes rJ_r = r\lambda (A \otimes J_r) + r(\lambda + 1)(A \otimes J_r)^t.
\]

Definition 3.14 Let \(T\) be a tournament with adjacency matrix \(A\). Then graph with adjacency matrix

\[
\begin{bmatrix}
0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & & 1 \\
\vdots & A & & A^t \\
0 & & 1 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & & 0 \\
\vdots & A^t & & A \\
1 & & 0
\end{bmatrix}
\]

is denoted by \(D(T)\).

In other words, if \(T = (V, E)\) is a tournament with \(V = \{x_1, \ldots, x_n\}\) then \(D(T)\) is the graph with vertex set \(\{v_0, v_1, \ldots, v_n, w_0, w_1, \ldots, w_n\}\) and arc set

\[
\{(v_0, v_i), (v_i, w_0), (w_0, w_i), (w_i, v_0) \mid i = 1, \ldots, n\} \cup \{(v_i, v_j), (v_j, w_i), (w_i, w_j), (w_j, v_i) \mid (x_i, x_j) \in E\}.
\]

Proposition 3.15 For any doubly regular tournament \(T\) of order \(m = 2k + 1 = 4\lambda + 3\), the graph \(D(T)\) is a doubly regular \((m + 1, 2)\)-team tournament.

Proof Let \(D\) be the adjacency matrix of \(D(T)\). Then an easy computation shows that

\[
D^2 = kD + kD^t + m(J - I - D - D^t).
\]  

(2)
3.3 Normally regular digraphs

The notion of a normally regular digraph was suggested by Jørgensen in [Jør] as a possible generalization of the notion of strongly regular graphs. For the reader’s convenience we provide here a short digest of some of the most important facts regarding this notion, in particular some facts about orientations of $\overline{m \circ K_r}$ that are normally regular digraphs.

**Definition 3.16** A directed graph $\Gamma$ with adjacency matrix $A$ is called a normally regular digraph if there exist numbers $k, \lambda, \mu$ such that

(i) $AA^t = kI + \lambda(A + A^t) + \mu(J - I - A - A^t)$, and

(ii) $A + A^t$ is a $\{0, 1\}$-matrix.

The condition (i) says that, for any pair $x, y$ of vertices, the number of common out neighbors of $x$ and $y$ is

\[
\begin{cases}
  k & \text{if } x = y, \\
  \lambda & \text{if } x \text{ and } y \text{ are adjacent, i.e. if either } (x, y) \text{ or } (y, x) \text{ is in } \Gamma, \\
  \mu & \text{if } x \text{ and } y \text{ are nonadjacent.}
\end{cases}
\]

In particular, every vertex has out-valency $k$, i.e., $AJ = kJ$. The second condition says that, for any pair $x, y$ of vertices, at most one of the arcs $(x, y)$ and $(y, x)$ is present in the graph.

One may consider general normally regular digraphs, satisfying condition (i) but not necessarily (ii). However in this paper it is natural to require condition (ii), as we investigate relations to other structures satisfying (ii).

It is convenient to introduce two new parameters

\[
\eta = k - \mu + (\mu - \lambda)^2 \quad \text{and} \quad \rho = k + \mu - \lambda.
\]

The matrix equation in the definition of normally regular digraphs can be written as follows

\[
(A + (\mu - \lambda)I)(A + (\mu - \lambda)I)^t = (k - \mu + (\mu - \lambda)^2)I + \mu J.
\]

Thus the matrix $B = (A + (\mu - \lambda)I)$ satisfies the following equations

\[
BB^t = \eta I + \mu J
\]

and (as $AJ = kJ$)

\[
BJ = \rho J.
\]

**Proposition 3.17 ([Jør])** If $A$ is an adjacency matrix of a normally regular digraph then $A$ is normal, i.e.,

\[
A^tA = AA^t.
\]

**Proof** It is sufficient to prove that $B$ is normal. Suppose first that $B$ is singular. Then one of the eigenvalues of $\eta I + \mu J$ is zero: $\eta = 0$ or $\eta + \mu v = 0$. Since $\mu, v \geq 0$ this is possible only when $\eta = k - \mu + (\mu - \lambda)^2 = 0$. This implies $k = \mu = \lambda$. But if $k > 0$ then the $k$ out-neighbors of a vertex span a $\lambda$-regular graph and so $k \geq 2\lambda + 1$. Thus $B$ is nonsingular.
From $BJ = \rho J$ we get $\rho^{-1}J = B^{-1}J$ and

$$B^t = B^{-1}(BB^t) = B^{-1}(\eta I + \mu J) = \eta B^{-1} + \mu \rho^{-1}J$$

(3)

Using that $J$ is symmetric, we get from this

$$\rho J = (BJ)^t = JB^t = \eta JB^{-1} + \mu \rho^{-1}J^2 = \eta JB^{-1} + \mu \rho^{-1}vJ.$$  

This implies that

$$JB^{-1} = \frac{\rho - \mu \rho^{-1}v}{\eta} J,$$

and so

$$vJ = J^2 = (JB^{-1})(BJ) = \frac{\rho - \mu \rho^{-1}v}{\eta} \rho vJ.$$

Thus

$$\frac{\rho - \mu \rho^{-1}v}{\eta} = \rho^{-1},$$

(4)

and $JB^{-1} = \rho^{-1}J$ or $\rho J = JB$. Now equation 3 implies

$$B^tB = \eta I + \mu \rho^{-1}JB = \eta I + \mu J = BB^t.$$  

Thus we have $A^tA = AA^t$.  

**Corollary 3.18** A normally regular digraph is a regular graph of valency $k$, i.e.,

$$AJ = JA = kJ$$

and the number of common in-neighbors of vertices $x$ and $y$ is equal to the number of common out-neighbors of $x$ and $y$.

Since the graph is regular we can now count pairs of vertices $(y, z)$ with $x \rightarrow y \leftarrow z$, $z \neq x$, where $x$ is a fixed vertex, in two ways. We get

$$k(k - 1) = 2k \lambda + (n - 1 - 2k)\mu,$$

which is equivalent to equation 4.

For any normally regular digraph we have $0 \leq \mu \leq k$. We next give a structural characterization of normally regular digraphs with equality in one of these inequalities.

**Theorem 3.19 ([Jor])** A directed graph $\Gamma$ is a normally regular digraph with $\mu = k$ if and only if $\Gamma$ is isomorphic to $C_s(T)$ for some doubly regular tournament $T$ and natural number $s$.

**Proof** It is easy to verify that if $T$ is a doubly regular tournament on $4t + 3$ vertices then $C_s(T)$ is a normally regular digraph with $(v, k, \lambda, \mu) = (s(4t + 3), s(2t + 1), st, s(2t + 1))$.

Suppose that $\Gamma$ is a normally regular digraph with $\mu = k$. Suppose that $x$ and $y$ are non-adjacent vertices in $\Gamma$. Then, as $\mu = k$, $N^+(x) = N^+(y)$. If $z$ is another vertex not adjacent to $x$ then $N^+(z) = N^+(x) = N^+(y)$. Since $\lambda \leq \frac{k-1}{2}$, $y$ and $z$ are non-adjacent. Non-adjacency is thus
a transitive relation on the vertex-set, which is therefore partitioned into classes, say \( V_1, \ldots, V_r \), such that any two vertices are adjacent if and only if they belong to distinct classes.

Suppose that \( x \in V_i \) dominates \( y \in V_j \), for some \( 1 \leq i, j \leq r \). Then every vertex \( x' \in V_i \) dominates every vertex \( y' \in V_j \). For if \( x \neq x' \) then \( x \) and \( x' \) are non-adjacent and have the same out-neighbors, so that \( x' \rightarrow y \), and similarly \( x' \rightarrow y' \). We also have \( 2k = |N^+(x)| + |N^-(x)| = v - |V_i| \). Thus \( \Gamma \) is isomorphic to \( C_s(T) \) for some regular tournament \( T \) and \( s = v - 2k \).

If \( v \rightarrow w \) is an arc in \( T \) then it follows that the number of common out-neighbors of \( v \) and \( w \) in \( T \) is \( \frac{k}{s} \). It follows from Lemma 3.2 that \( T \) is a doubly regular tournament.

**Theorem 3.20 ([Jor])** A connected graph \( \Gamma \) is a normally regular digraph with \( \mu = 0 \) if and only if \( \Gamma \) is isomorphic to \( T \) or \( D(T) \) for some doubly regular tournament \( T \), or \( \Gamma \) is a directed cycle.

**Proof** It is easy to verify that if \( \Gamma \) is a doubly regular tournament \( T \) or a directed cycle then \( \Gamma \) is a normally regular digraph with \( \mu = 0 \). (Since a doubly regular tournament has no pairs of non-adjacent vertices, \( \mu \) is arbitrary but we may choose \( \mu = 0 \).)

Suppose that \( \Gamma \) is isomorphic to \( D(T) \) for some doubly regular tournament \( T \) with adjacency matrix \( A \) as in definition 3.14. If the order of \( T \) is \( s = 2\ell + 1 \) then, by Lemma 3.2, the adjacency matrix \( D \) of \( D(T) \)

\[
DD^t = \begin{bmatrix}
  s & \ell & \ldots & \ell & 0 & \ell & \ldots & \ell \\
  \ell & \ell & & & & & & \\
  : & \ell J + (\ell + 1)I & : & \ell (A + A^t) & & & & \\
  \ell & \ell & & & & & & \\
  0 & \ell & \ldots & \ell & s & \ell & \ldots & \ell \\
  \ell & \ell & & & & & & \\
  : & \ell (A + A^t) & : & \ell J + (\ell + 1)I & & & & \\
  \ell & \ell & & & & & & 
\end{bmatrix} = sI + \ell (D + D^t). \tag{6}
\]

Thus \( \Gamma \) is a normally regular digraph with parameters \( (v, k, \lambda, \mu) = (2s + 2, s, \ell, 0) \).

Let \( \Gamma \) be a connected normally regular digraph with \( \mu = 0 \). Suppose that \( \Gamma \) is not a directed cycle. Then \( k \geq 2 \). If \( \Gamma \) is a tournament then it follows from Lemma 3.2 that \( T \) is a doubly regular tournament. So we may assume that \( \Gamma \) is not a tournament. Let \( x \) be a vertex in \( \Gamma \) and let \( y \) be a vertex not adjacent to \( x \).

Since \( \mu = 0 \), it follows from equation 5 that \( \lambda = \frac{k-1}{2} \geq 1 \). Thus the subgraphs spanned by \( N^+(x) \) and \( N^-(x) \) and \( \lambda \)-regular tournaments. Since \( \mu = 0 \), \( y \) has no out-neighbor in \( N^+(x) \). But since \( \Gamma \) is connected, we may assume that \( y \) has an out-neighbor in \( N^-(x) \). Let \( U = N^+(y) \cap N^-(x) \) and suppose that \( U \neq N^-(x) \). Since \( N^-(x) \) is a regular tournament, there exists vertices \( u \in U \) and \( u' \in N^-(x) \setminus U \) so that \( u' \rightarrow u \). But \( y \) and \( u' \) have a common out-neighbor and so they are adjacent. So we have \( y \rightarrow u' \), a contradiction. Thus \( N^+(y) = N^-(x) \).

Suppose there is another vertex \( y' \) non-adjacent to \( x \) so that \( y' \) has an out-neighbor in \( N^-(x) \). Then as above \( N^+(y') = N^-(x) \). But then \( y \) and \( y' \) have \( k \) common out-neighbors, a contradiction to \( k > \lambda > \mu \neq 0 \). Thus, as \( N^-(x) \) is \( \lambda \)-regular, any vertex in \( u \in N^-(x) \) has \( k - 1 - \lambda = \lambda \) in-neighbors in \( N^+(x) \). It also has \( \lambda \) out-neighbors in \( N^+(x) \), as \( x \) and \( u \) have \( \lambda \) common out-neighbors.
Now there is a vertex \( w \in N^+(x) \) so that \( w \) and \( y \) have a common out-neighbor in \( N^-(x) \). Thus \( w \to y \). As above, it follows that \( N^-(y) = N^+(x) \) and that any vertex in \( N^+(x) \) has \( \lambda \) out-neighbors and \( \lambda \) in-neighbors in \( N^-(x) \).

Since \( \Gamma \) is connected, it has vertex set \( \{x, y\} \cup N^+(x) \cup N^-(x) \). For every vertex there is a unique non-adjacent vertex. Thus \( \Gamma \) is a \((\frac{v}{2}, 2)\)-team tournament. In fact, we have now proved that it is a doubly regular \((\frac{v}{2}, 2)\)-team tournament with \( \alpha = \beta = \lambda \) and \( \gamma = k \). (Since \( x \) is an arbitrary vertex we need only consider paths of length 2 starting at \( x \).) The statement now follows from Theorem 4.6. ■

### 3.4 Association schemes and their relation graphs

**Definition 3.21** Let \( X \) be finite set and let \( \{R_0, R_1, \ldots, R_d\} \) be a partition of \( X \times X \). Then we say that \( \mathcal{X} = (X, \{R_0, R_1, \ldots, R_d\}) \) is a \( d \)-class association scheme if the following conditions are satisfied

- \( R_0 = \{(x, x) \mid x \in X\} \),
- for each \( i \in \{0, \ldots, d\} \) there exists \( i' \in \{0, \ldots, d\} \) such that \( R_{i'} = \{(x, y) \mid (y, x) \in R_i\} \),
- for each triple \( (i, j, h) \), \( i, j, h \in \{0, \ldots, d\} \) there exists a number \( p_{ij}^h \) such that for all \( x, y \in X \) with \((x, y) \in R_h\) there are exactly \( p_{ij}^h \) elements \( z \in X \) so that \((x, z) \in R_i\) and \((z, y) \in R_j\).

The binary relations \( R_1, \ldots, R_d \) can be identified with (arc sets of) graphs on \( X \). A relation \( R_i \) with \( i' = i \) is an undirected graph and a relation \( R_i \) with \( i' \neq i \) is a directed graph. If the graphs \( R_1, \ldots, R_d \) all are connected then we say that \( \mathcal{X} \) is primitive, otherwise it is imprimitive.

If \( i = i' \) for all \( i \) then \( \mathcal{X} \) is said to be symmetric, otherwise it is nonsymmetric. For a symmetric 2-class association scheme the graph \( R_1 \) is a strongly regular graph and \( R_2 \) is its complement. Conversely, if \( \Gamma \) is a strongly regular graph then \( \Gamma \) and \( \overline{\Gamma} \) (i.e., \( E(\Gamma) \) and \( E(\overline{\Gamma}) \)) form the relations of a symmetric 2-class association scheme. It is known that for an imprimitive symmetric 2-class association scheme, either \( R_1 \) or \( R_2 \) is isomorphic to \( m \circ K_r \) for some \( m, r \).

For a nonsymmetric 2-class association scheme the graph \( R_1 \) is a doubly regular tournament and conversely (see Lemma 3.2) for every doubly regular tournament \( R_1 \) there is a nonsymmetric 2-class association scheme \( (X, \{R_0, R_1, R_2\}) \). Every nonsymmetric 2-class association scheme is primitive.

Let \( X = \{x_1, \ldots, x_n\} \) be a finite set and let \( R_0, \ldots, R_d \) be graphs (possibly with loops) with vertex set \( X \). Let \( A_i \) be the adjacency matrix of \( R_i \), for \( i = 0, \ldots, d \). Let \( A \) be the vector space spanned by \( A_0, \ldots, A_d \). Then \( \{R_0, \ldots, R_d\} \) is a partition of \( X \times X \) if and only if \( A_0 + \cdots + A_d = J \). The other conditions in definition 3.21 are equivalent to

- \( A_0 = I \),
- \( A_i^t = A_{i'} \),
- \( A \) is closed under matrix multiplication and \( A_i A_j = \sum_h p_{ij}^h A_h \).
Furthermore, if these conditions are satisfied then $\mathcal{A}$ is closed under the Schur-Hadamard product. The algebra $\mathcal{A}$ with usual matrix product and Schur-Hadamard product is called the adjacency algebra of the association scheme. The association scheme is symmetric if and only if $\mathcal{A}$ consists of symmetric matrices.

An association scheme $\mathcal{X}$ is called commutative if $p_{ij}^h = p_{ji}^h$ for all $i, j, h$. Thus $\mathcal{X}$ is commutative if and only if $\mathcal{A}$ is commutative. The adjacency algebra of a commutative association scheme is usually called the Bose-Mesner algebra of this scheme. Every symmetric association scheme is commutative, for if $A$ and $B$ belong to an algebra of symmetric matrices then $AB = A^tB^t = (BA)^t = BA$. Higman [Hig75] proved that every $d$-class association scheme with $d \leq 4$ is commutative.

For a commutative nonsymmetric association scheme $\mathcal{X} = (X, \{R_0, R_1, \ldots, R_d\})$ it is known (see [Ban84], Remark in Section 2.2) that a symmetric association scheme $\tilde{\mathcal{X}}$ can be constructed by replacing each pair $R_i, R_{i'}$ ($i \neq i'$) by $R_i \cup R_{i'}$. $\tilde{\mathcal{X}}$ is called the symmetrization of $\mathcal{X}$. The corresponding Bose-Mesner algebras satisfy $\tilde{A} = \{A \in \mathcal{A} \mid A = A^t\}$.

Let $\mathcal{X}$ be a nonsymmetric 3-class association scheme. Then $\mathcal{X}$ consists of a pair of nonsymmetric relations $R_1, R_2$ and one symmetric relation $R_3$ (in addition to $R_0$). By the above mentioned theorem of Higman, $\mathcal{X}$ is commutative. Thus the symmetrization of $\mathcal{X}$ exists and is a 2-class association scheme with relations $R_1 \cup R_2$ and $R_3$.

A nonsymmetric association scheme $\mathcal{X}$ is imprimitive if and only if the symmetrization $\tilde{\mathcal{X}}$ is imprimitive. Now suppose that $\mathcal{X}$ is imprimitive. Then $R_1$ is an orientation of either $m \circ K_r$ or $m \circ K_r$.

**Proposition 3.22** $R_1$ is an orientation of $m \circ K_r$ if and only if $R_1$ consists of $m$ disjoint doubly regular tournaments of order $r$.

**Proof** Suppose first that $R_1$ is an orientation of $m \circ K_r$. Then $R_1$ consists of $m$ disjoint tournaments of order $r$. $R_1$ and each of the tournaments are regular of valency $p_{12}^1$. Let $T$ be one of the components of $R_1$. Let $(x, y) \in T$. Then there are $p_{11}^1$ paths of length 2 from $x$ to $y$. Thus $T$ is a doubly regular tournament.

Conversely, if $R_1$ is a disjoint union of $m$ doubly regular tournaments of order $r$ (not necessarily isomorphic) then $(V(R_1), \{R_0, R_1, R_2, R_3\})$ is an imprimitive 3-class association scheme with $R_2 = R_1$ and $R_3 = R_1 \cup R_2$. ■

We now consider the case where $R_1$ is an orientation of $m \circ K_r$. First we give two types of examples.

**Proposition 3.23** Let $R_1$ be one of the graphs $\mathcal{C}_r(T)$ or $\mathcal{D}(T)$ where $T$ is a doubly regular tournament and $r \in \mathbb{N}$. Let $R_2 = R_1$ and $R_3 = R_1 \cup R_2$. Then $(V(R_1), \{R_0, R_1, R_2, R_3\})$ is an imprimitive 3-class association scheme.

**Proof** Let $T$ be a doubly regular tournament on $v$ vertices and with adjacency matrix $A$. Then $\mathcal{C}_r(T)$ has adjacency matrix $A \otimes J_r$. The relations $R_2$, $R_3$ and $R_0$ in this case have adjacency matrices $A^t \otimes J_r$, $I_v \otimes (J_r - I_r)$ and $I_v \otimes I_r$. It follows from Lemma 2.1 and Lemma 3.2 that the set of linear combinations of these matrices is closed under multiplication.
Suppose now that $D$ is the adjacency matrix of $D(T)$, for some doubly regular tournament $T$. From equations 2 and 6 we know that $D^2 = \ell D + \ell D^t = (D^t)^2$ and $DD^t = sI + \ell(D + D^t)$ for some $s = 2\ell + 1$. By Proposition 3.17, $D^t D = DD^t$. We also know that $JD = DJ = JD^t = D^t J = sJ$.

It follows that any product of two matrices in $B = \{ I, D, D^t, J \}$ is a linear combination of matrices in $B$. Thus the relations with adjacency matrices in $B$ form an imprimitive association scheme with 3 classes. ■

In section 5 we will continue our analysis of links between orientations of $m \circ K_r$ and association schemes based on a characterization of doubly regular $(m, r)$-team tournaments which will be achieved in the next section.

4 Characterization of doubly regular $(m, r)$-team tournaments

4.1 Three types of graphs

Let $\Gamma$ be a doubly regular $(m, r)$-team tournament and let $\alpha, \beta, \gamma$ be as in definition 3.12. Let $V(\Gamma) = V_1 \cup \ldots \cup V_m$ be the partition of the vertex set into $m$ independent sets of size $r$.

In this section we prove that $\Gamma$ is one of three types:

Type I. $C_r(T)$ for some doubly regular tournament $T$.

Type II. Every vertex in $V_i$ dominates exactly half of the vertices in each $V_j$, for $j \neq i$. This type includes $D(T)$ for a doubly regular tournament $T$.

Type III. Every vertex in $V_i$ dominates either all vertices of $V_j$, exactly half of the vertices in each $V_j$ or none of the vertices of $V_j$, for $j \neq i$, but $\Gamma$ is not one of the above types. (No examples of this structure is known.)

More details about the structure of the last two types is given below.

For $x \in V_i$, let $d_j(x) = |N^+(x) \cap V_j|$ be the number of out-neighbors of $x$ in $V_j$.

Lemma 4.1 Let $x \in V_i$ and $y \in V_j$ and suppose that $(x, y) \in E(\Gamma)$. Then

$$d_j(x) - d_i(y) = \beta - \alpha.$$

Proof The vertex $x$ has $k - d_j(x)$ out-neighbors outside $V_j$. Exactly $\alpha$ of these out-neighbors are in-neighbors of $y$ and $k - d_j(x) - \alpha$ vertices are common out-neighbors of $x$ and $y$. Similarly, $y$ has $k - d_i(y)$ out-neighbors outside $V_i$. $\beta$ of these vertices are in-neighbors of $x$ and $k - d_i(y) - \beta$ vertices are common out-neighbors of $x$ and $y$. Thus $k - d_j(x) - \alpha = k - d_i(y) - \beta$. This proves the lemma. ■

Lemma 4.2 For each pair $(i, j)$, $i \neq j$, either

1) there exists a constant $c_{ij}$ so that $d_j(x) = c_{ij}$ for every $x \in V_i$ or else

2) $V_i$ is partitioned into two nonempty sets $V_i = V'_i \cup V''_i$ so that all arcs are directed from $V'_i$ to $V_j$ and from $V_j$ to $V''_i$. 15
Proof Suppose that the constant $c_{ij}$ does not exist. Let $x', x'' \in V_i$ be such that $d_j(x') \neq d_j(x'')$. Let $y \in V_j$. If $(x', y), (x'', y) \in E(\Gamma)$ then
\[
d_j(x') - d_i(y) = \beta - \alpha = d_j(x'') - d_i(y),
\]
i.e., $d_j(x') = d_j(x'')$, a contradiction. If $(y, x'), (y, x'') \in E(\Gamma)$ then
\[
d_i(y) - d_j(x') = \beta - \alpha = d_i(y) - d_j(x''),
\]
a contradiction. Thus $V_j$ is partitioned into two sets $V_j' = V_j' \cup V_j''$ (one of the sets may be empty) so that $V_j'$ is the set of out-neighbors of $x'$ in $V_j$ and $V_j''$ is the set of out-neighbors of $x''$ in $V_j$. For every vertex $x \in V_i$ with an out-neighbor in $V_j'$, $d_j(x) = d_j(x') \neq d_j(x'')$. Thus $x$ has no out-neighbor in $V_j''$ and so the set of out-neighbors of $x$ in $V_j$ is exactly $V_j'$. Similarly, if $x$ has an out-neighbor in $V_j''$ then the set of out-neighbors of $x$ in $V_j$ is $V_j''$. If $x$ has no out-neighbor in $V_j$ then $x$ has a common in-neighbor with either $x'$ or $x''$, say with $x''$. Then $d_j(x'') = d_j(x) = 0$ and so the set of out-neighbors of $x$ in $V_j$ is $V_j'' = \emptyset$. Thus we also get a partition $V_i = V_i' \cup V_i''$ so that the arcs between $V_i$ and $V_j$ are directed from $V_i'$ to $V_j'$, from $V_i'$ to $V_j''$, from $V_i''$ to $V_j'$ and from $V_i''$ to $V_j''$. If either $V_j'$ or $V_j''$ is empty then the second case of the lemma holds. So suppose that $V_j'$ and $V_j''$ are both nonempty. Let $y' \in V_j'$ and $y'' \in V_j''$. From
\[
|V_j'| - |V_j''| = d_j(x'') - d_i(y') = \beta - \alpha = d_i(y'') - d_j(x') = |V_j''| - |V_j'|,
\]
we get by adding $|V_i'| + |V_i''| = r = |V_j'| + |V_j''|$ that $d_j(x') = |V_j''| = |V_j'| = d_j(x'')$, a contradiction. 

\[\square\]

Theorem 4.3 Let $\Gamma$ be a doubly regular $(m, r)$-team tournament. Then $\Gamma$ satisfies one of the following.

Type I. $\beta - \alpha = r$ and $\Gamma$ is isomorphic to $C_r(T)$, for some doubly regular tournament $T$.

Type II. $\beta - \alpha = 0$, $r$ even and $d_i(x) = \frac{r}{2}$ for all $x \notin V_i$.

Type III. $\beta - \alpha = \frac{r}{2}$ and for every pair $(i, j)$ either $V_i$ is partitioned into two sets $V_i'$ and $V_i''$ of size $\frac{r}{2}$ so that all arcs between $V_i$ and $V_j$ are directed from $V_i'$ to $V_j$ and from $V_j$ to $V_i''$ or similarly with $i$ and $j$ interchanged.

Proof Suppose that for some given pair $(i, j)$ case 2 of Lemma 4.2 is satisfied. Let $x' \in V_i', x'' \in V_i'', y \in V_j$. By Lemma 4.1,
\[
|V_j| - |V_i''| = d_j(x') - d_i(y) = \beta - \alpha = d_i(y'') - d_j(x') = |V_i''|.
\]
Thus $|V_i''| = \frac{1}{2}|V_j| = \frac{r}{2}$, $|V_i'| = \frac{r}{2}$ and $\beta - \alpha = \frac{r}{2}$. Similarly, if case 2 is satisfied for the pair $(j, i)$.

Suppose that there exist constants $c_{ij}$ and $c_{ji}$ so that $d_j(x) = c_{ij}$ for every $x \in V_i$ and $d_i(x) = c_{ji}$ for every $x \in V_j$. Then there are two possibilities.

a) One of $c_{ij}, c_{ji}$ is 0, suppose that $c_{ji} = 0$ and $c_{ij} = r$. Then $\beta - \alpha = c_{ij} - c_{ji} = r$.

b) $c_{ij}$ and $c_{ji}$ are both positive, so that there exist arcs from $V_i$ to $V_j$ and arcs from $V_j$ to $V_i$. Then $c_{ij} - c_{ji} = \beta - \alpha = c_{ji} - c_{ij}$, i.e., $c_{ij} = c_{ji} = \frac{r}{2}$ and $\beta - \alpha = 0$.
Since the value of $\beta - \alpha$ is independent of the choice of $\{i, j\}$, we see that if case 2 of Lemma 4.2 is satisfied for at least one pair $(i, j)$ then $\Gamma$ is Type III, and if b) is satisfied for some pair $(i, j)$ then $\Gamma$ is Type II.

Suppose now that a) holds for every pair $\{i, j\}$. Then clearly $\Gamma$ is isomorphic to $C_r(T)$ for some regular tournament $T$. Suppose $(x, y) \in E(T)$ and that there are $\lambda$ paths from $x$ to $y$ of length 2 in $T$. Then there are exactly $r\lambda$ paths of length 2 between vertices in $\Gamma$ in the cocliques corresponding to $x$ and $y$. Thus $\lambda = \frac{\alpha}{r}$ is constant and so $T$ is doubly regular.

For doubly regular $(m, r)$-team tournaments of Type II and Type III it is possible to compute the parameters $\alpha, \beta, \gamma$ from $m$ and $r$ and to give some restrictions on $m$ and $r$.

**Theorem 4.4** Let $\Gamma$ be a doubly regular $(m, r)$-team tournament of Type II. Then

1) $\alpha = \beta = \frac{(m-2)r}{4}$,

2) $\gamma = \frac{(m-1)r^2}{4(r-1)}$, and

3) $r - 1$ divides $m - 1$.

**Proof**

1) Let $x \in V_1$. The number of directed paths of length 2 from $x$ to a vertex $y \notin V_1$ is $(m-1)^2 \cdot (m-2)$. Since the number of such $y$ is $(m-1)r$, so $\alpha = \beta = \frac{(m-1)^2(m-2)}{(m-1)r} = \frac{(m-2)r}{4}$.

2) The number of directed paths of length 2 from $x$ to a vertex $y \in V_1$ is $(m-1)^2 \cdot r$. The number of such vertices $y$ is $r - 1$. Thus $\gamma = \frac{(m-1)^2}{r-1} = \frac{(m-1)r^2}{4(r-1)}$.

3) Since $\gamma$ is an integer and $r$ and $r - 1$ are relatively prime, $r - 1$ divides $m - 1$.

**Theorem 4.5** Let $\Gamma$ be a doubly regular $(m, r)$-team tournament of Type III. Then

1) $\alpha = \frac{(m-1)r}{4} - \frac{3r}{8}$,

2) $\beta = \frac{(m-1)r}{4} + \frac{r}{8}$,

3) $\gamma = \frac{(m-1)r^2}{8(r-1)}$,

4) for every $i$ and for every $x \in V_i$, the sets $\{j \mid d_j(x) = 0\}$, $\{j \mid d_j(x) = \frac{r}{2}\}$ and $\{j \mid d_j(x) = r\}$ have cardinality $\frac{m-1}{4}$, $\frac{m-1}{2}$ and $\frac{m-1}{4}$, respectively.

5) $8$ divides $r$, and

6) $4(r-1)$ divides $m - 1$.

**Proof** For each pair $\{i, j\}$ either $(i, j)$ or $(j, i)$ satisfy case 2 of Lemma 4.2. Thus in the subgraph spanned by $V_i \cup V_j$ there are $r^3$ directed paths of length 2 joining two nonadjacent vertices. The total number of directed paths of length 2 joining two nonadjacent vertices in $\Gamma$ is $\left(\frac{m}{2}\right)^3$. The number of ordered pairs of nonadjacent vertices is $mr(r-1)$. Thus $\gamma = \frac{(m-1)r^3}{6(r-1)}$ as
\[ \binom{m}{2} r^2 = mr(r-1)\gamma. \] Since the number of directed paths of length 2 starting at a vertex \( x \) is independent of \( x \), \( \Gamma \) satisfies property 4.

The number of arcs in \( \Gamma \) is \( mrk \) and the number of directed paths of length 2 is \( mrk^2 \).

Thus

\[
mrk^2 - \left( \frac{m}{2} \right) \frac{r^3}{4} = mrk(\alpha + \beta).
\]

Since \( k = \frac{(m-1)r}{2} \),

\[
\frac{(m-1)^2 r^2}{4} - \frac{(m-1)r^2}{8} = \frac{(m-1)r}{2}(\alpha + \beta),
\]

i. e.,

\[
\frac{(m-1)r}{2} - \frac{r}{4} = \alpha + \beta.
\]

Since \( \beta - \alpha = \frac{r}{8} \), we get the required solution for \( \alpha \) and \( \beta \).

Since \( \beta \) and \( \frac{m-1}{4} \) are integers, \( \frac{r}{8} \) is also an integer.

Since \( \gamma \) is an integer and \( r - 1 \) and \( r \) are relatively prime, \( r - 1 \) divides \( m - 1 \). Since 4 divides \( m - 1 \) and \( r - 1 \) is odd, \( 4(r - 1) \) divides \( m - 1 \). \( \blacksquare \)

### 4.2 Doubly regular \((m, 2)\)-team tournaments

We are mainly interested in the case \( r = 2 \). In this case Song [So95] proved that either \( m \equiv 0 \mod 4 \) or \( m \equiv 3 \mod 4 \).

**Theorem 4.6** Let \( \Gamma \) be a doubly regular \((m, 2)\)-team tournament. Then \( \Gamma \) is isomorphic to either \( C_2(T) \) or \( D(T) \) for some doubly regular tournament \( T \).

**Proof** Suppose that \( \Gamma \) is not isomorphic to \( C_2(T) \) for any doubly regular tournament \( T \). Then, by Theorem 4.5, \( \Gamma \) is of Type II. By Theorem 4.3, every vertex in \( V_i \) has exactly one out-neighbor in \( V_j \), \( i \neq j \). Thus the subgraph spanned by \( V_i \cup V_j \) is a directed 4-cycle.

Let \( v_0 \) be a vertex in \( \Gamma \) and \( v_1, \ldots, v_k \) be the out-neighbors of \( v_0 \). For \( i = 0, \ldots, k \) let \( w_i \) be the unique vertex in \( \Gamma \) not adjacent to \( v_i \). Then \( V(\Gamma) = \{v_0, \ldots, v_k, w_0, \ldots, w_k\} \) and these vertices are distinct.

Suppose that \((v_i, v_j) \in E(\Gamma)\). Since the subgraph spanned by \( \{v_i, v_j, w_i, w_j\} \) is a 4-cycle, we also have \((v_j, w_i), (w_i, w_j), (w_j, v_i) \in E(\Gamma)\). Thus \( \Gamma \) is isomorphic to \( D(T) \) where \( T \) is the tournament spanned by \( N^+(v_0) \). Since there are \( \alpha \) paths of length 2 from \( v_0 \) to \( v_1 \), \( v_i \) has in-valency \( \alpha \) in \( T \). Thus \( T \) is regular of valency \( \alpha \).

Suppose that \((v_i, v_j) \in E(T)\). Let \( \lambda \) denote the number of paths of length 2 in \( T \) from \( v_i \) to \( v_j \). Then \( v_i \) and \( v_j \) have \( \alpha - 1 - \lambda \) common out-neighbors in \( T \). And so \( v_j \) has \( \alpha - (\alpha - 1 - \lambda) = \lambda + 1 \) out-neighbors in \( T \) which are in-neighbors of \( v_i \), i.e. there are \( \lambda + 1 \) paths of length 2 in \( T \) from \( v_j \) to \( v_i \).

Any path of length 2 from \( v_i \) to \( v_j \) contained in \( \Gamma \) but not in \( T \) has the form \( v_i w_i v_j \). But \( v_i w_i v_j \) is a directed path if and only if \( v_j v_i v_j \) is a directed path from \( v_j \) to \( v_i \). Thus \( \Gamma \) contains \( \lambda + (\lambda + 1) \) paths of length 2 from \( v_i \) to \( v_j \). Since this number is \( \alpha \), \( \lambda = \frac{m-1}{2} \) is constant, and so \( T \) is doubly regular. \( \blacksquare \)
Remark. In the definition of $D(T)$ (Definition 3.14) the vertex $v_0$ plays special role. But the above proof shows that if $T$ is a doubly regular tournament then any vertex in $D(T)$ can be chosen to play this role. Thus for any vertex $x$, the subgraph spanned by $N^+(x)$ is a doubly regular tournament. This tournament need not be isomorphic to $N^+(v_0)$, but Hadamard matrices constructed from them in Theorem 3.11 are equivalent. The relation between tournaments $N^+(v_0)$ and $N^+(x)$ was investigated in [Jør94].

5 Equivalence of main structures

We will begin with a simple corollary of some results which were obtained in previous sections.

Corollary 5.1 Every doubly regular $(m, r)$-team tournament with $r = 2$ is a relation of an imprimitive 3-class association scheme.

Proof This follows from Theorem 4.6 and Proposition 3.23. □

Proposition 5.2 A doubly regular $(m, r)$-team tournament of Type III is not a relation of a 3-class association scheme.

Proof Suppose that $R_1$ is a doubly regular $(m, r)$-team tournament of Type III and that \{$R_0, R_1, R_2, R_3$\} are the relations of a 3-class association scheme, where $R_2 = R_1^t$. Then $R_3$ is an undirected graph with components $V_1, \ldots , V_m$, each of which span a complete graph. Let $(x, y) \in R_1$. Then there exist $i \neq j$ such that $x \in V_i$ and $y \in V_j$. Then in $R_1$, $x$ has $p_{13}^1 + p_{10}^1$ out-neighbors in $V_j$. But by Theorem 4.5, this number is not a constant. □

Proposition 5.3 Let $(X, \{R_0, R_1, R_2, R_3\})$ be a nonsymmetric imprimitive 3-class association scheme such that $R_1$ and $R_2$ are directed graphs and $R_3$ is a disconnected undirected graph. Then $R_1$ is a doubly regular $(m, r)$-team tournament of Type I or Type II, for some $m, r \in \mathbb{N}$.

Proof Let $A_0, A_1, A_2, A_3$ be the adjacency matrices of $R_0, R_1, R_2, R_3$. Then $A_i^2 = p_{11}^1 A_0 + p_{11}^2 A_1 + p_{11}^3 A_2 + p_{11}^3 A_3 = p_{11}^1 A_1 + p_{11}^2 A_2 + p_{11}^3 (J - I - A_1 - A_1^t)$, as $p_{11}^0 = 0$. Thus $R_1$ is a doubly regular team tournament. By Proposition 5.2, it is Type I or Type II. □

Therefore, the imprimitive nonsymmetric 3-class association schemes for which $R_1$ is an orientation $m \circ K_r$ for $r = 2$, are characterized by $R_1$ being isomorphic to either $C_2(T)$ or $D(T)$ for some doubly regular tournament $T$. For $r \geq 3$, we have the following theorem by Goldbach and Claasen [GolC96]. For completeness we give an alternative proof of this statement here.

Theorem 5.4 Let $(X, \{R_0, R_1, R_2, R_3\})$ be a nonsymmetric imprimitive 3-class association scheme such that $R_1$ is an orientation $m \circ K_r$. Then either $R_1$ is isomorphic to $C_r(T)$ for some doubly regular tournament $T$ or else

- $r - 1$ divides $m - 1$ and
• \( r \) and \( m \) are both even.

**Proof** If \( R_1 \) is not isomorphic to \( C_r(T) \) then by Proposition 5.3, \( R_1 \) is a doubly regular \((m,r)\)-team tournament of Type II. It follows from Theorem 4.3 and Corollary 4.4 that \( r \) is even and that \( r - 1 \) divides \( m - 1 \). In Corollary 5.8 below, it will be proved that \( m \) is even. \( \blacksquare \)

We have proved above that a nonsymmetric relation of an imprimitive 3-class association scheme is a doubly regular team tournament of Type I or Type II. The converse is also true.

**Theorem 5.5** Let \( R_1 \) be a doubly regular \((m,r)\)-team tournament of Type I or Type II. Let \( R_2 = R_1^t \) and \( R_3 = R_1 \cup R_2 \). Then \((V(R_1), \{R_0, R_1, R_2, R_3\})\) is an imprimitive 3-class association scheme.

We know from Proposition 3.23 that Theorem 5.5 is true for doubly regular \((m,r)\)-team tournaments of Type I. In order to prove the theorem for doubly regular \((m,r)\)-team tournaments of Type II we need the following lemma.

**Lemma 5.6** Let \( A \) be the adjacency matrix of a doubly regular \((m,r)\)-team tournament \( \Gamma \) of Type II. Then \( A \) is normal.

**Proof** We must prove that \( AA^t = A^t A \), i.e., that for every pair \( x, y \) of vertices, the number of common out-neighbors is equal to the number of common in-neighbors. Since \( \Gamma \) is regular, this is true for \( x = y \). Suppose that \( x \) and \( y \) are distinct. By Theorem 4.3, \( x \) and \( y \) both have exactly \( \frac{r}{2} \) out-neighbors in each \( V_i \), where \( x, y \notin V_i \). Thus for each \( V_i \) the number of common out-neighbors of \( x \) and \( y \) in \( V_i \) is equal to the number of common in-neighbors of \( x \) and \( y \) (these are the vertices in \( V_i \) that out-neighbors of neither \( x \) nor \( y \).) \( \blacksquare \)

It follows from the Spectral Theorem for normal matrices that \( A \) has an orthogonal diagonalization. Let \( \lambda_0, \ldots, \lambda_s \) be the distinct eigenvalues. Let \( E_0, \ldots, E_s \) be the orthogonal projections on the corresponding eigenspaces. Then

\[
A = \sum_{i=0}^{s} \lambda_i E_i,
\]

and \( I = \sum_{i=0}^{s} E_i \). Then

\[
A^2 = \sum_{i=0}^{s} \lambda_i^2 E_i
\]

and since orthogonal projections are self-adjoint,

\[
A^t = \sum_{i=0}^{s} \lambda_i E_i
\]

We may assume that \( \lambda_0 = k = \frac{(m-1)r}{2} \). Then \( J = mr E_0 \).
Lemma 5.7 Let \( \Gamma \) be a doubly regular \((m, r)\)-team tournament of Type II. Then the adjacency matrix \( A \) of \( \Gamma \) has exactly four distinct eigenvalues.

Proof If we use the above expressions for \( A, A^2 \) and \( A^4 \) in the equation
\[
A^2 = \alpha A + \beta A^t + \gamma (J - I - A - A^t)
\]
and multiply by \( E_j, j \neq 0 \) then we get
\[
\lambda_j^2 = \alpha \lambda_j + \beta \overline{\lambda_j} + \gamma (-1 - \lambda_j - \overline{\lambda_j}).
\]
Let \( \lambda_j = a + bi \), where \( a \) and \( b \) are real. Since \( \beta = \alpha \),
\[
a^2 - b^2 + 2abi = 2a\alpha + \gamma (-1 - 2a).
\]
Since the imaginary part of this is \( 2ab = 0 \), either \( b = 0 \) (and \( \lambda_i \) is real) or \( a = 0 \) and \( b^2 = \gamma \).
The adjacency matrix of \( m \circ K_r \) is
\[
A + A^t = \sum_{i=0}^4 (\lambda_i + \overline{\lambda_i})E_i.
\]
Thus \( \lambda_j + \overline{\lambda_j} = 2a \) is one of the three eigenvalues \( 2k = (m - 1)r, 0, -r \) of \( m \circ K_r \). Thus the eigenvalues of \( \Gamma \) are \( k, -\frac{r}{2}, \pm i\sqrt{\gamma} \).

Corollary 5.8 Let \( \Gamma \) be a doubly regular \((m, r)\)-team tournament of Type II. Then \( m \) is even.

Proof Since the multiplicity of 0 as an eigenvalue of \( m \circ K_r \) is \( m(r - 1) \), the eigenvalues \( \pm i\sqrt{\gamma} \) of \( \Gamma \) both have multiplicity \( \frac{m(r - 1)}{2} \), \( m \) must be even.

Proof of Theorem 5.5 Let \( A \) be the adjacency matrix of \( R_1 \). By the above lemma, \( A \) has four eigenvalues \( \lambda_0, \ldots, \lambda_3 \) with corresponding orthogonal projections \( E_0, \ldots, E_3 \). Let \( \mathcal{A} \) denote the set of matrices that are linear combinations of \( E_0, \ldots, E_3 \). Then \( \mathcal{A} \) is a four dimensional algebra closed under matrix multiplication.
\[ B = \{ I, A, A^t, J - I - A - A^t \} \] is a set of four linearly independent matrices and by the above remarks they are contained in \( \mathcal{A} \). Thus \( B \) is a basis of \( \mathcal{A} \) and any product of matrices from \( B \) is a linear combination of \( B \). The matrices in \( B \) are adjacency matrices of relations \( R_0, R_1, R_2, R_3 \) of an imprimitive 3-class association scheme.

6 Vertex-transitive doubly regular team tournaments

In [Jør94] it was proved that if \( T \) is either the Paley tournament \( P_q \) (cf. Theorem 3.6), or the Szekeres tournament \( S_q \) then \( D(T) \) is vertex transitive. We now consider groups acting transitively on a graph \( D(T) \) for some doubly regular tournament \( T \). It is a well known fact that a tournament does not have any automorphism of order 2. A modification of the proof of this shows that \( D(T) \) has only one automorphism of order 2.
Lemma 6.1 Let \( \Gamma \) be the graph \( D(T) \) for some tournament \( T \). Then the automorphism group of \( \Gamma \) contains a unique involution \( \phi \). This involution \( \phi \) maps a vertex \( x \) to the unique vertex in \( \Gamma \) nonadjacent to \( x \).

**Proof** It is easy to verify that \( \phi \) is an automorphism. Let \( \psi \) be an involutory automorphism of \( \Gamma \). If for some vertex \( x \), \((x, \psi(x)) \in E(\Gamma) \) then \((\psi(x), \psi^2(x)) = (\psi(x), x) \in E(\Gamma) \), a contradiction. Similarly, \((\psi(x), x) \in E(\Gamma) \) is not possible. If \( \psi(x) = x \) then for any vertex \( y \in N^+(x) \), \( \psi(y) = y \), as \( N^+(x) \) is a tournament. Then it follows from the structure of \( \Gamma \) that \( \phi(y) = y \) for every vertex \( y \in V(\Gamma) \). Thus for every vertex \( x \), \( \psi(x) \) is the unique vertex not adjacent to \( x \) and \( \psi = \phi \). ■

Lemma 6.2 Let \( G \) be a group of automorphisms of \( \Gamma = D(T) \), for some doubly regular tournament \( T \), and suppose that \( G \) has order a power of 2. Then the action of \( G \) on \( V(\Gamma) \) is semiregular.

**Proof** Let \( x \in V(\Gamma) \). By Lemma 6.1, \( G_x \) has odd order, but the order of \( G_x \) is a power of 2. Thus \( |G_x| = 1 \). ■

We wish to prove that the Sylow 2-subgroup of the automorphism group of a vertex transitive graph \( D(T) \) is a generalized quaternion group. Ito [Itô94] showed that the Sylow 2-subgroup of an Hadamard group can not be cyclic (see Section 8 below). We prove a similar theorem for the automorphism group of \( D(T) \), but we do not require that the graph is a Cayley graph. We first need a few lemmas.

Lemma 6.3 Let \( \Gamma \) be a directed graph with vertex set \( \{x_0, \ldots, x_{4m-1}\} \) such that

- the permutation \( g = (x_0, \ldots, x_{4m-1}) \) is an automorphism;
- the pairs of non-adjacent vertices are the pairs \( \{x_i, x_{i+2m}\} \), for \( i = 0, \ldots, 2m-1 \); and
- if \( x_0 \to x_\ell \) then \( x_\ell \to x_{2m} \).

Then the number of common out-neighbors of \( x_0 \) and \( x_1 \) in \( \Gamma \) is even.

**Proof** If \( x_0 \to x_\ell \) then, since \( g^{2m} \) is an automorphism, \( x_0 \to x_\ell \to x_{2m} \to x_{2m+i} \to x_0 \) and by applying \( g^{-\ell} \) to this we get \( x_0 \to x_{2m-\ell} \to x_{2m} \to x_{4m-\ell} \to x_0 \).

The proof is by induction on the number \( s \) of out-neighbors of \( x_0 \) in \( \{x_{2m+1}, \ldots, x_{4m-1}\} \). If \( s = 0 \) then \( N^+(x_0) = \{x_1, \ldots, x_{2m-1}\} \); and so, \( N^+(x_1) = \{x_2, \ldots, x_{2m}\} \). Thus \( x_0 \) and \( x_1 \) have \( 2m - 2 \) common out-neighbors.

For the induction step, suppose that for some \( 1 \leq \ell \leq m \) the arcs in the set \( B = \{x_i \to x_{i+\ell}, x_i \to x_{i+2m-\ell} \mid i = 0, \ldots, 4m - 1\} \) are reversed.

Suppose first that \( 1 < \ell < m \). If \( x_0 \to x_{\ell-1} \) and thus \( x_1 \to x_\ell \), \( x_0 \to x_{2m-\ell+1} \), \( x_0 \to x_{-\ell+1} \), and \( x_1 \to x_{2m+\ell} \), then by reversal of arcs in \( B \) the vertices \( x_\ell \) and \( x_{2m-\ell+1} \) are removed from the set of common out-neighbors of \( x_0 \) and \( x_1 \). If \( x_0 \to x_{\ell-1} \) then \( x_{-\ell+1} \) and \( x_{2m+\ell} \) will be new common out-neighbors of \( x_0 \) and \( x_1 \) after reversal of arcs in \( B \). If \( x_0 \to x_{\ell+1} \) and thus \( x_1 \to x_{2m-\ell} \), \( x_0 \to x_{2m+\ell+1} \), and \( x_1 \to x_{-\ell} \), then reversal will remove \( x_{\ell+1} \) and \( x_{2m-\ell} \) from the
list of common out-neighbors, but if \( x_0 \leftarrow x_{\ell+1} \) then \( x_{2m+\ell+1} \) and \( x_{-\ell} \) will be new common out-neighbors. Thus the number of common out-neighbors of \( x_0 \) and \( x_1 \) will increase by \( \pm 2 \pm 2 \).

The case \( \ell = 1 \) is essentially the same as the above except that arcs \( x_0 \to x_{\ell-1} \) and \( x_0 \to x_{2m-\ell+1} \) can not exist. The number of common out-neighbors will increase by \( \pm 2 \).

In the case \( \ell = m \), \( x_0 \to x_{\ell-1} \) exists if and only if \( x_0 \to x_{\ell+1} \) exists. The number of common out-neighbors will increase by \( \pm 2 \).

In each case the number of common out-neighbors of \( x_0 \) and \( x_1 \) will remain even. \( \blacksquare \)

**Lemma 6.4** Let \( \Gamma \) be a directed graph with vertex set \( \{x_0, \ldots, x_{4m-1}, v_0, \ldots, v_{4m-1}\} \) such that

- the permutation \( g = (x_0, \ldots, x_{4m-1})(v_0, \ldots, v_{4m-1}) \) is an automorphism,
- for each \( i = 0, \ldots, 2m-1 \), either \( x_0 \to v_i \) or \( x_0 \leftarrow v_i \), and
- if \( x_0 \to v_l \) then \( v_{l+2m} \to x_0 \).

Then the number of common out-neighbors of \( x_0 \) and \( x_1 \) in \( D = \{v_0, \ldots, v_{4m-1}\} \) is odd.

**Proof** The proof is by induction on the number \( s \) of out-neighbors of \( x_0 \) in \( \{v_{2m}, \ldots, v_{4m-1}\} \).

If \( s = 0 \) then \( x_0 \) is adjacent to \( v_0, \ldots, v_{2m-1} \) and since \( g \) is an automorphism, \( x_1 \) is adjacent to \( v_1, \ldots, v_{2m} \). Thus the number of common out-neighbors of \( x_0 \) and \( x_1 \) in \( D \) is \( 2m - 1 \).

For the induction step, suppose that for some \( l \), the arcs \( x_i \to v_{i+l} \) and thus \( v_{i+l+2m} \to x_i \), \( i = 0, \ldots, 4m-1 \) are reversed. This reversal only affects common out-neighbors in \( \{v_l, v_{l+1}, v_{l+2m}, v_{l+2m+1}\} \). Thus the number of common out-neighbors of \( x_0 \) and \( x_1 \) in \( D \) increases by \( 2, 0, \) or \( -2 \), if \( x_0 \) was adjacent to none of \( x_{l-1} \) or \( x_{l+1} \), to exactly one of them, or to both of them, respectively.

Thus \( x_0 \) and \( x_1 \) always have an odd number of common out-neighbors in \( D \). \( \blacksquare \)

**Lemma 6.5** Let \( \Gamma \) be isomorphic to \( D(T) \) for some doubly regular tournament \( T \). Let \( g \) be a semiregular automorphism of \( \Gamma \). Then the number of orbits of \( g \) is even.

**Proof** Suppose that the number of orbits is odd. Then 8 divides the order of \( g \). Let the order of \( g \) be \( 4m \). Then \( g^{2m} \) is the unique automorphism of order 2. Let \( x_0 \in V(\Gamma) \) and let \( C \) be the orbit containing \( x_0 \). Let \( x_i = x_0g^i \). Then the subgraph of \( \Gamma \) spanned by \( C \) satisfies the conditions in Lemma 6.3. Thus the number of common out-neighbors of \( x_0 \) and \( x_1 \) in \( C \) is even.

Let \( D = \{v_0, \ldots, v_{4m-1}\} \) be any other orbit of \( g \) with \( v_i = v_0g^i \). The subgraph spanned by \( C \) and \( D \) satisfies the condition of Lemma 6.4. Thus the number of common out-neighbors of \( x_0 \) and \( x_1 \) in \( D \) is odd. Since there is an even number of orbits different from \( C \), \( x_0 \) and \( x_1 \) have an even number of common out-neighbors outside \( C \) and thus an even number of common out-neighbors in \( \Gamma \). However, the number of common out-neighbors of two adjacent vertices in \( D(T) \) is odd, a contradiction. \( \blacksquare \)

**Theorem 6.6** Suppose that for some doubly regular tournament \( T \), the graph \( \Gamma = D(T) \) is vertex transitive and has automorphism group \( G \). Then a Sylow 2-subgroup \( S \) of \( G \) is the generalized quaternion group of order \( 2^n \), where \( 2^n \) is the highest power of 2 that divides the order of \( \Gamma \).
Since $\Gamma$ is vertex transitive the order of $G$ is divisible by the order of $\Gamma$ and thus by $2^n$. It follows that the order of $S$ is at least $2^n$. Since the action of $S$ on $V(\Gamma)$ is semiregular, the order of $S$ is not divisible by $2^{n+1}$. Thus $|S| = 2^n$.

Since $G$ and thus $S$ has a unique involution and the order of $S$ is a power of 2, $S$ is either a cyclic group or a generalized quaternion group, see [Gor68]. Since $S$ is semiregular, the number of orbits under the action of $S$ is $|\Gamma|/2^n$, which is odd. Thus by Lemma 6.5, $S$ cannot be cyclic.

**Corollary 6.7** Suppose that for some doubly regular tournament $T$, the graph $\Gamma = D(T)$ is vertex transitive and has order $2^n$. Then $\Gamma$ is a Cayley graph of the generalized quaternion group of order $2^n$.

**Proof** By Theorem 6.6, a Sylow 2-subgroup $S$ of the automorphism group of $\Gamma$ is the generalized quaternion group of order $2^n$. By Lemma 6.2, $S$ is semiregular and thus regular. ■

**7 Cayley graphs and group rings**

Many normally regular digraphs arise from Cayley graphs, group rings and difference sets. We consider first under what condition on the connection set $S \subset G$, the Cayley graph $\text{Cay}(G, S)$ becomes a normally regular digraph.

**Proposition 7.1** Let $G$ be a group with identity 1 and let $S \subset G$. Let $S^{(-1)}$ denote the set $\{s^{-1} | s \in S\}$. Let $\Gamma$ be $\text{Cay}(G, S)$. Then $\Gamma$ is a normally regular digraph with parameters $(|G|, |S|, \lambda, \mu)$ if and only if $S \cap S^{(-1)} = \emptyset$ and for any $g \in G$ the number of pairs $(s, t) \in S \times S$ such that $st^{-1} = g$ is

$$
\begin{cases}
  k & \text{if } g = 1, \\
  \lambda & \text{if } g \in S \cup S^{(-1)}, \\
  \mu & \text{otherwise}.
\end{cases}
$$

**Proof** The condition $S \cap S^{(-1)} = \emptyset$ ensures that the graph does not have undirected edges nor loops. Let $x$ and $y$ be distinct vertices in $\Gamma$. There exists a unique $g \in G$ so that $xg = y$. Then $x \rightarrow y$ if $g \in S$ and $y \rightarrow x$ if $g \in S^{(-1)}$. Let $z$ be a vertex in $\Gamma$. Then $x \rightarrow z$ if $z = xs$, for some $s \in S$, and $y \rightarrow z$ if $z = yt$, for some $t \in S$. Thus $z$ is a common out-neighbor of $x$ and $y$ if and only if $xs = yt = xgt$. This is equivalent to $g = st^{-1}$, and so the number of common out-neighbors of $x$ and $y$ is equal to the number of pairs $(s, t) \in S \times S$ so that $g = st^{-1}$. ■

For a (multiplicative) group $G$, the group ring $\mathbb{Z}G$ is the set of formal sums $\sum_{g \in G} c_g g$, where $c_g \in \mathbb{Z}$. Then $\mathbb{Z}G$ is a ring with sum

$$
\left(\sum_{g \in G} c_g g\right) + \left(\sum_{g \in G} d_g g\right) = \sum_{g \in G} (c_g + d_g)g
$$

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and product
\[
\left( \sum_{g \in G} c_g g \right) \cdot \left( \sum_{g \in G} d_g g \right) = \sum_{g \in G} \left( \sum_{h \in G} c_h d_{h^{-1} g} \right) g.
\]

For a set \( S \subseteq G \) we define \( S = \sum_{g \in S} g \in \mathbb{Z}G \). We write \( \{ g \} \) as \( g \).

The condition in Proposition 7.1 can be reformulated in terms of the group ring.

**Corollary 7.2** \( \text{Cay}(G, S) \) is a normally regular digraph with parameters \((v, k, \lambda, \mu)\) where \( v = |G| \) and \( k = |S| \) if and only if

- \( S \cap S^{(-1)} = \emptyset \), and
- \( S \cdot S^{(-1)} = k1 + \lambda(S + S^{(-1)}) + \mu(G - S - S^{(-1)} - 1) \) in the group ring \( \mathbb{Z}G \).

For the particular case \( \mu = 0 \), we get the following corollary.

**Corollary 7.3** \( \text{Cay}(G, S) \) is isomorphic to \( D(T) \) for some doubly regular tournament \( T \) if and only if

- \( |G| = 4\lambda + 4 \) and \( |S| = 2\lambda + 1 \), for some \( \lambda \),
- \( S \cap S^{(-1)} = \emptyset \), and
- \( S \cdot S^{(-1)} = (2\lambda + 1)1 + \lambda(S + S^{(-1)}) \).

The condition \( S \cap S^{(-1)} = \emptyset \) implies that \( 1 \notin S \). We will next consider a property of sets satisfied by \( S \cup \{ 1 \} \).

**Definition 7.4** Let \( G \) be a group of order \( mn \). Let \( N \) be a subgroup of \( G \) of index \( m \). A subset \( D \subset G \) is said to be a relative \((m, n, k, \lambda)\) difference set with forbidden subgroup \( N \) if \( |D| = k \) and for any \( g \in G \) the number of pairs \((s, t) \in D \times D \) such that \( g = st^{-1} \) is exactly

\[
\begin{cases} 
  k & \text{if } g = 1, \\
  0 & \text{if } g \in N, \ g \neq 1, \\
  \lambda & \text{if } g \notin N.
\end{cases}
\]

Usually it is assumed that \( N \) is a normal subgroup. Since \( N \) is a forbidden subgroup, \( D \) has at most one element in each (right) coset of \( N \). We consider the case where \( m = k \); that is, \( D \) has exactly one element in each coset of \( N \).

The condition in the definition of relative difference sets can be stated in terms of the group ring.

**Lemma 7.5** Let \( N < G \) be as in the above definition. Then a subset \( D \subset G \) with \( |D| = k \) is a relative difference set if and only if

\[
D \cdot D^{(-1)} = k1 + \lambda(G - N).
\]
In the next lemma we see that from one relative difference set several new relative difference sets can be constructed by shifting $D$.

**Lemma 7.6** If $D \subset G$ is a relative difference set then for any $g \in G$, $Dg = \{dg \mid d \in D\}$ is a relative difference set with the same forbidden subgroup and the same parameters.

**Proof**

\[
Dg \cdot (Dg)^{(-1)} = Dg \cdot g^{-1}D^{(-1)} = D \cdot D^{(-1)}.
\]

\[
\square
\]

In Lemma 6.1 we have observed that for any tournament $T$ the graph $D(T)$ has a unique automorphism of order 2. This automorphism is forbidden as a “difference” of elements in the connection set if $D(T)$ is a Cayley graph.

**Proposition 7.7** Suppose that $\Gamma = \text{Cay}(G, S)$ is isomorphic to $D(T)$ for some doubly regular tournament $T$ and $|G| = 4\lambda + 4$. Then $S \cup \{1\}$ is a relative $(2n, 2, 2n, n)$ difference set in $G$ with forbidden subgroup $N = \langle \phi \rangle$, where $\phi$ is the unique involutory automorphism of $\Gamma$ and $n = \lambda + 1$.

**Proof** From Corollary 7.3 we get

\[
(S \cup \{1\} \cdot (S \cup \{1\})^{-1} = S \cdot S^{(-1)} + S^{(-1)} + 1 = (2\lambda + 2)1 + (\lambda + 1)(S + S^{(-1)}) = 2n \cdot 1 + n(S \cup S^{(-1)}).
\]

By Lemma 6.1, $S \cup S^{(-1)} = G \setminus N$. Thus we have

\[
(S \cup \{1\} \cdot (S \cup \{1\})^{-1} = 2n \cdot 1 + n(G - N).
\]

\[
\square
\]

Suppose that $D \subset G$ is relative $(2n, 2, 2n, n)$ difference set. Then, by shifting if necessary, we may assume that $1 \in D$.

**Proposition 7.8** If $D \subset G$ is a relative $(2n, 2, 2n, n)$ difference set with forbidden subgroup $N = \langle \phi \rangle$ such that $D \cap D^{(-1)} = \{1\}$ then $\text{Cay}(G, D \setminus \{1\})$ is isomorphic to $D(T)$ for some doubly regular tournament $T$.

**Proof** Let $S = D \setminus \{1\}$. Since $D \cap D^{(-1)} = \{1\}$, we get $S \cap S^{(-1)} = \emptyset$,

\[
D + D^{(-1)} = G - \emptyset + 1 = G - N + 2 \text{ and } S \cup S^{(-1)} = G \setminus N. \text{ Thus }
\]

\[
S \cdot S^{(-1)} = (D - 1) \cdot (D^{(-1)} - 1) = D \cdot D^{(-1)} - D - D^{(-1)} + 1 = 2n \cdot 1 + n(G - N) - (G - N + 2) + 1 = (n - 1)(G - N) + (2n - 1) \cdot 1 = (n - 1)(S + S^{(-1)}) + (2n - 1) \cdot 1. \text{ If we let } \lambda = n - 1 \text{ then we get condition in Corollary 7.3.} \]

\[
\square
\]

### 8 Ito’s Conjecture and S-rings over dicyclic groups

In [Ito94], N. Ito proposed to construct Hadamard matrices from relative difference sets and he introduced the following definition.

**Definition 8.1** A group of order $4n$ is called an Hadamard group if it contains a relative $(2n, 2, 2n, n)$ difference set, relative to some normal subgroup of order 2.
Ito [Ito94] proved that the existence of an Hadamard group of order $4n$ implies the existence of an Hadamard matrix of order $2n$.

**Theorem 8.2** Let $G$ be a group of order $4n$ and let $N = \langle u \rangle$ be a normal subgroup of order 2. Let $D \subset G$ be a relative $(2n, 2n, n)$ difference set. Let $a_1, \ldots, a_{2n}$ be representatives of the cosets of $N$. Let $H = (h_{ij})$ be the $2n \times 2n$ matrix with

$$h_{ij} = \begin{cases} -1 & \text{if } a_j \in a_i D, \\ 1 & \text{if } ua_j \in a_i D. \end{cases}$$

Then $H$ is an Hadamard matrix.

**Proof** First, we must prove that $h_{ij}$ is well defined for every $i, j$, i.e., that $|\{a_j, ua_j\} \cap a_i D| = 1$. Suppose that $\{a_j, ua_j\} \subset a_i D$. Then there exists $d_1, d_2 \in D$ so that $a_j = a_i d_1$ and $ua_j = a_i d_2$ and so $d_1 d_2^{-1} = (a_i^{-1} a_j)(a_i^{-1} ua_j)^{-1} = a_i^{-1} a_j a_i^{-1} u^{-1} a_i = a_i^{-1} ua_i = u$, as $N$ is a normal subgroup. But this is a contradiction, as $N$ is the forbidden subgroup. Thus $|\{a_j, ua_j\} \cap a_i D| \leq 1$. Since $a_i D$ has cardinality $2n$ and there are $2n$ sets $\{a_j, ua_j\}$, $|\{a_j, ua_j\} \cap a_i D| = 1$ for every $i, j$.

Since $H$ has entries $\pm 1$, we only need to prove that $HH^t = 2nI$. For $1 \leq i, \ell \leq n$, $i \neq \ell$, the cardinality of the set $a_i D \cap a_\ell D$ is

$$|\{g \in G : g = a_i d_1 \text{ and } g = a_\ell d_2, \text{for some } d_1, d_2 \in D\}|$$

$$= |\{(d_1, d_2) \in D \times D : a_i d_1 = a_\ell d_2\}|$$

$$= |\{(d_1, d_2) \in D \times D : di_2^{-1} = a_i^{-1} a_\ell\}|$$

$$= n,$$

as $D$ is a relative $(2n, 2n, n)$ difference set and $a_i^{-1} a_\ell \notin N$. The number of columns $j$ such that $h_{ij} = h_{ij} = -1$ is $|\{j : a_j \in a_i D \cap a_\ell D\}|$, and the number of columns $j$ where $h_{ij} = h_{ij} = 1$ is $|\{j : ua_j \in a_i D \cap a_\ell D\}|$. Since $G = \{a_1, \ldots, a_{2n}, ua_1, \ldots, ua_{2n}\}$, the number of values of $j$ where $h_{ij}$ and $h_{ij}$ are equal is $|\{a_i D \cap a_\ell D\}| = n$. And $h_{ij}$ and $h_{ij}$ are different for the remaining $n$ values of $j$. Thus the dot product of rows $i$ and $\ell$ is 0. \hfill \blacksquare

Ito [Ito94] also proved that the Sylow 2-subgroup of an Hadamard group is not cyclic or dihedral. In [Ito97] he conjectured that every dicyclic group of order $8t$, for some $t$, is an Hadamard group. By Theorem 8.2 this would imply the Hadamard matrix conjecture. Schmidt [Sch99] proved that this is true for $t \leq 46$. Ito [Ito94] also investigated conditions for the constructed Hadamard matrix to be skew.

**Theorem 8.3** Let $G$ be Hadamard group of order $4n$ and let $N = \langle u \rangle$ be a normal subgroup of order 2. Let $D \subset G$ be a relative $(2n, 2n, n)$ difference set and let $H = (h_{ij})$ be the Hadamard matrix constructed from $D$ in Theorem 8.2. Then $H$ is a skew Hadamard matrix if and only if $D \cap D^{(1)} = \{1\}$.

**Proof** A diagonal entry $h_{ii}$ is $-1$ if and only if $1 \in D$.

Suppose that $d \neq 1$ satisfies that $d, d^{-1} \in D$. There exists indices $i \neq j$ and $u' \in N$ so that $a_i \in N$ and $u' a_j = d$. Then $a_j = (u' a_j) a_d = a_d^2 = 1$, and $a_i = (a_i u') a_j d^{-1}$. The value of $h_{ij}$ depends on $u' a_i$ and the value of $h_{ji}$ depends on $a_i u' = u' a_i$. Thus $h_{ij} = h_{ji}$. \hfill 27
Suppose next that \( D \cap D^{(-1)} = \{1\} \). Then for \( g \in G \setminus N \), \( g \in D \) if and only if \( g^{-1} \notin D \). For \( i \neq j \), \( h_{ij} = -1 \iff a_i^{-1} a_j \in D \iff a_j^{-1} a_i \notin D \iff h_{ji} = 1 \). ■

Ito [Ito94] called a group with the property of Theorem 8.3 a skew Hadamard group. From Corollary 7.3, Proposition 7.8 and Theorem 8.3 we get the following corollary.

**Corollary 8.4** A group is a skew Hadamard group if and only if it is regular subgroup of the automorphism group of \( D(T) \), for some doubly regular tournament \( T \).

In view of Ito’s conjecture we consider the following problem.

**Problem 8.5** For which value of \( t \), is the dicyclic group of order \( 8t \) a skew Hadamard group?

In what follows, we consider this problem in detail. In particular, we prove that if \( 4t \) is a prime power then the dicyclic group of order \( 8t \) is a skew Hadamard group. A computer search has shown that there are no skew Hadamard groups of order 72, see also [ItoO96].

Ito [Ito94] showed that the Hadamard matrices corresponding to Paley tournaments can be constructed from skew Hadamard groups. We state this result in terms of \( D(P) \).

**Theorem 8.6** Let \( q \equiv 3 \mod 4 \) be a prime power and \( P \) be the Paley tournament of order \( q \). Let \( \Gamma = D(P) \). Then \( SL(2, q) \) acts as a group of automorphisms on \( \Gamma \) and this group contains a dicyclic subgroup acting regularly on \( V(\Gamma) \).

The proof uses the following basic property of the group \( SL(2, q) \) of \( 2 \times 2 \) matrices over \( \mathbb{F}_q \) with determinant 1.

**Lemma 8.7** \( SL(2, q) \) has order \( q(q - 1)(q + 1) \) and it contains a subgroup isomorphic to the dicyclic group of order \( 2(q + 1) \).

**Proof** Since \( SL(2, q) \) is isomorphic to \( SU(2, q^2) \) (see [Hup67], Hilfssatz II.8.8), it suffices to find a subgroup of \( SU(2, q^2) \) isomorphic to the dicyclic group. Let \( \lambda \in \mathbb{F}_q^* \) be an element of order \( q + 1 \). Let

\[
x = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^q \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Then \( x, y \in SU(2, q^2) \), \( x^{q+1} = y^2, y^4 = 1, xy = y \) and so \( x \) and \( y \) generate a dicyclic subgroup of \( SU(2, q^2) \) of order \( 2(q + 1) \). ■

**Proof of Theorem 8.6** Let \( \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\} \) be the multiplicative group of the finite field of order \( q \). Let \( Q \) be the set of nonzero squares in \( \mathbb{F}_q \) and let \( N = \mathbb{F}_q^* \setminus Q \). Then \( Q \) is a subgroup of \( \mathbb{F}_q^* \) of index 2 and \( N = -Q \), as \(-1 \notin Q \). Let \( V = \mathbb{F}_q^2 \) be the 2-dimensional vector space of row vectors over \( \mathbb{F}_q \). We define a relation \( \sim \) on \( V \setminus \{0\} \) by

\[
v \sim w \text{ if } v = \alpha w \text{ for some } \alpha \in Q.
\]

This relation is an equivalence relation and we have a partition of \( V \setminus \{0\} \) in equivalence classes. This relation partitions each 1-dimensional subspace into two classes. The class containing
$v = (x, y)$ will be denoted by $[v] = [x, y]$. We denote by $X = (V \setminus \{0\}) / \sim$ the set of all equivalence classes. The number of classes is $|X| = \frac{q^2-1}{(q-1)^2} = 2(q+1)$.

If $v = aw$ for $v, w \in V \setminus \{0\}$ and $a \in Q$ then $vA = awA$ for any matrix $A \in GL(2, q)$. Thus the natural action of $GL(2, q)$ on $V \setminus \{0\}$ by right multiplication preserves the relation $\sim$.

For any $[\alpha, \beta] \in X$, the matrix \( \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \) or if $\alpha = 0$ the matrix \( \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \) has determinant 1 and maps $[1, 0]$ to $[\alpha, \beta]$. It follows that the special linear group $G = SL(2, q) \equiv \{ g \in GL(2, q) \mid \det(g) = 1 \}$ acts transitively on $X$ by right multiplication. If some matrix $g \in G$ fixes every $[v] \in X$ then $g = \lambda I$ for some $\lambda \in Q$. Since $\det(g) = \lambda^2 = 1$ and $\lambda \in Q$, $\lambda = 1$. Thus $G$ acts faithfully on $X$.

The stabilizer of $[1, 0]$ is $G_{[1, 0]} = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix} \mid \alpha \in Q, \gamma \in \mathbb{F}_q \right\}$.

$G$ has an action on $X \times X$ defined by $(x, y)^g = (xg, yg)$ for all $x, y \in X$ and $g \in G$. Let $E$ be the orbit in this action containing $([1, 0], [0, 1])$ and let $\Gamma$ be the graph with vertex set $X$ and arc set $E$. We need to prove that $\Gamma$ is isomorphic to $D(P_q)$.

The set of out neighbors of $[1, 0]$ is the orbit containing $[0, 1]$ under the action of $G_{[1, 0]}$ on $X$.

The orbit consists of elements of the form $[\gamma, \alpha^{-1}] = [\beta, 1]$ (where $\beta = \alpha \gamma$).

Since the matrix \( \begin{pmatrix} \beta & -1 \\ 1 - \beta^2 & \beta \end{pmatrix} \) is the automorphism \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) maps $([1, 0], [\beta, 1])$ to $([-1, 0], [ \beta, 1])$ the in-neighbors of $[1, 0]$ are the elements $[\beta, -1]$, $\beta \in \mathbb{F}_q$.

Thus $[-1, 0]$ is a unique vertex not adjacent to $[1, 0]$ and the automorphism \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) in $G$ shows that $[-1, 0]$ has in-neighbors $[\beta, 1]$ and out-neighbors $[\beta, -1]$.

Any automorphism $g \in G$ mapping $[1, 0]$ to $[v] \in X$ also maps $[-1, 0]$ to the unique vertex $[v]$ not adjacent to $[v]$. Since $G$ acts transitively on arcs of $\Gamma$ (by definition of $E = E(\Gamma)$), on $E^1 = \{(x, y) \mid (y, x) \in E\}$ and on pairs of non adjacent vertices, $\Gamma$ is a doubly regular $(q+1, 2)$-team tournament. Since there are $q$ paths of length 2 from $[1, 0]$ to $[-1, 0]$, $\Gamma$ is isomorphic to $D(N^+([1, 0]))$, by Theorem 4.6.

Let $[\alpha, 1], [\beta, 1] \in N^+([1, 0])$. An automorphism $g \in G$ maps $([1, 0], [0, 1])$ to $([\alpha, 1], [\beta, 1])$ if and only if $g = \begin{pmatrix} \delta \alpha & \delta \\ \epsilon \beta & \epsilon \end{pmatrix}$, where $\delta, \epsilon \in Q$. Since $\det(g) = (\alpha - \beta)\delta\epsilon$, such $\delta, \epsilon$ exist if and only $\alpha - \beta \in Q$. Thus $(\alpha, 1), (\beta, 1) \in E(\Gamma)$ if and only $\alpha - \beta \in Q$. By Lemma 3.5, $N^+([1, 0])$ is isomorphic to the Paley tournament $P_q$ and so $\Gamma$ is isomorphic to $D(P_q)$.

It follows, from Lemma 8.7, that there is a subgroup $H < G = SL(2, q)$ isomorphic to the dihedral group of order 2$(q+1)$, and that the stabilizer $G_{[1, 0]}$ in $G$ of $[1, 0]$ has order \( \frac{q(q-1)(q+1)}{2(q+1)} \), as $|X| = 2(q+1)$. Thus the order of the stabilizer $H_{[1, 0]}$ in $H$ of $[1, 0]$ divides $\frac{1}{2}q(q-1)$. It also divides $|H| = 2(q+1)$. Since $\frac{1}{2}q(q-1)$ and $2(q+1)$ are relatively prime, $H_{[1, 0]} = 1$. Since $|H| = |X|$, $H$ acts regularly on $X$. 

**Example.** Other groups than the dihedral groups may appear as regular subgroups of the automorphism groups of graphs $D(T)$. We show some examples found with the use the
By Theorem 8.6, the automorphism group of $D(P_q)$ has a regular group isomorphic to the dicyclic group of order $2q + 2$. However, for some values of $q$ there are some additional regular subgroups.

The automorphism group of $D(P_{11})$ has two conjugate classes of regular subgroups. One class consists of dicyclic groups of order 24. The other class consists of groups isomorphic to $\langle x, y, z \mid x^2 = y^2, y^4 = z^3 = 1, xyx = y, zx = yz, zy = x \rangle \simeq SL(2, 3)$.

The automorphism group of $D(P_{23})$ has three conjugate classes of regular subgroups. One class consists of dicyclic groups of order 48. The other two classes consist of groups isomorphic to $\langle x, y, z, w \mid x^2 = y^2 = w^2, y^4 = z^3 = 1, xyx = y, zx = yz, zy = x \rangle \simeq SL(2, 3)$.

The automorphism group of $D(P_{59})$ also has three conjugate classes of regular subgroups. One class consists of dicyclic groups of order 120. The other two classes consist of groups isomorphic to “SmallGroup(120,5) in GAP”. (URL: http://???)

Let $S_q$ be the doubly regular tournament of order $2q + 1$ obtained by applying Theorem 3.6 to the Paley tournament $P_q$, for a prime power $q \equiv 3 \mod 4$, and let $G$ be the automorphism group of $S_q$. Then it was proved in [Jør94] that $G$ acts transitively on the vertices. Computer experiments show that for small value of $q$ (at least up to $q = 47$) $G$ has a regular subgroup isomorphic to the dicyclic group.

For $q = 11$ there is one additional regular subgroup (of order 48) isomorphic to $\langle x, y, z, w \mid x^2 = y^2 = w^2, y^4 = z^3 = 1, xyx = y, zx = yz, zy = x \rangle$.

There is a tournament $T$ of order 23 so that $D(T)$ is a Cayley graph of the group $G = \langle x, y, z \mid x^6 = y^2, x^3 = z^2, y^4 = 1, xyx = y, zx = x^5z, zy = x^3yz \rangle$. $G$ is the full group of automorphisms of $D(T)$, i.e., $T$ has a trivial automorphism group.

9 Concluding remarks
References


[Jør] Normally regular digraphs.


