Exact controllability of a multilayer Rao-Nakra beam with minimal number of boundary controls

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Outline of talk

1. Multilayer Rao-Nakra Model
   - Model description
   - Equations of motion; semigroup formulation

2. Spectrum
   - Riesz basis property
   - Stability

3. Controllability results
   - Moment problem
   - Scalar boundary control
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Multilayer Rao-Nakra beam

1. Multilayer “sandwich beam” structure with $n = 2m + 1$ layers
2. $m + 1$ “stiff layers”, $m$ compliant layers
3. No-slip on interfaces
4. piecewise linear displacements w.r.t. thickness
5. all shear occurs in compliant layers
6. independent dynamics for each layer (“discrete layer theory”)
7. viscous damping in shear for compliant layers
Equations of motion:

\[ m \ddot{w} - \alpha D_x^2 \dot{w} + KD_x^4 w - D_x N^T h_\varepsilon (G \phi + \tilde{G} \dot{\phi}) = 0 \quad \text{on} \ (0, L) \times (0, \infty) \]

\[ h_\phi p_\phi \ddot{v}_\phi - h_\phi E_\phi D_x^2 v_\phi + B^T (G \phi + \tilde{G} \dot{\phi}) = 0 \quad \text{on} \ (0, L) \times (0, \infty) \]

where \( \phi = h_\varepsilon^{-1} B v_\phi + N w_x \).

- \( w \): transverse displacement
- \( \phi \): vector of shear angle of compliant layers
- \( v_\phi \): vector of longitudinal displacement of stiff layers

Physical constants: \( m \): mass density, \( \alpha \): moment of inertia; \( K \) bending stiffness; \( p_\phi \) density matrix for stiff layers; \( h_\phi \) thickness matrix for stiff layers; \( h_\varepsilon \) thickness matrix for compliant layers; \( E_\phi \) stiffness matrix; \( G \) shear stiffness matrix and \( \tilde{G} \) shear viscosity matrix; \( B \) dimensionless matrix; \( N \) dimensionless vector
In addition we consider the following controlled boundary conditions:

\[
\begin{align*}
    w(0, t) &= D_x^2 w(0, t) = D_x v_\mathcal{O}(0, t) = w(L, t) = 0 \quad t > 0, \\
    D_x^2 w(L, t) &= M(t), \quad D_x v_\mathcal{O}(L, t) = g_\mathcal{O}(t) \quad t > 0.
\end{align*}
\]

Boundary control functions: \(M(t)\), the applied moment, and \(g_\mathcal{O}(t) = (g_1(t), g_3(t), \ldots, g_n(t))^T\), the longitudinal force.

**Remark:** One generally expects that one control is needed for each equation to obtain an exactly controllable system. For example, this is true with the undamped system.

**Goal:** choose parameters \(\mathbf{G}, \tilde{\mathbf{G}}\) appropriately, so that the system is exactly controllable with \(M\) and \(g_\mathcal{O}\) taken as a linear function of a lower dimensional control function \(\mathbf{u}\).
Semigroup formulation

Let \((u, v) = \int_0^L u \cdot \bar{v} \, dx\), and define quadratic forms \(a\) and \(c\) by

\[
c(w, v_O) = (mw, w) + \alpha(w_x, w_x) + (h_O p_O v_O, v_O)
\]

\[
a(w, v_O) = K(w_{xx}, w_{xx}) + (h_O E_O v_{Ox}, v_{Ox}) + (G h_O \epsilon \varphi, \varphi).
\]

The energy of the beam is given by

\[
\mathcal{E}(t) = \frac{1}{2} (c(\dot{w}, \dot{v}_O) + a(w, v_O))
\]
Let $U = (u, u^T := (w, v)_O^T$, $V = (v, v)^T := (\dot{w}, \dot{v}_O)^T$, $Y = (U, V)$. Also define $J$ by $J\theta = m\theta - \alpha D_x^2 \theta$ and

$$A_1 U = \begin{pmatrix} J^{-1}(-KD_x^4 u + D_x N^T G[(Bu + h\varepsilon ND_x u)] \\ h_O^{-1} p_O^{-1} [h_O E_O D_x^2 u - B^T G[h^{-1}_\varepsilon (Bu + h\varepsilon ND_x u)]] \end{pmatrix}.$$

and

$$A_2 V = \begin{pmatrix} J^{-1}(D_x Nh\varepsilon \tilde{G}[h^{-1}_\varepsilon (Bv + h\varepsilon ND_x v)] \\ h_O^{-1} p_O^{-1} [-B^T \tilde{G}[h^{-1}_\varepsilon (Bv + h\varepsilon ND_x v)]] \end{pmatrix}.$$

$$B\{M, g_O\} = \begin{pmatrix} J^{-1} M(t) \delta_L'(x) \\ h_O^{-1} p_O^{-1} g_O \delta_L(x) \end{pmatrix}.$$
The control problem can be reformulated as follows:

\[
\frac{d}{dt} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ B\{M, g_o\} \end{pmatrix}.
\]

The finite energy space is \( X_1 = H^2(0, L) \cap H^1_0(0, L) \times (H^1(0, L))^{m+1} \) for position and \( X_0 = H^1_0(0, L) \times (L^2(0, L))^{m+1} \) for velocity.

**Theorem 1.** If the initial data \( \{U_0, V_0\}^T \) belongs to the finite energy space \( X_1 \times X_0 \) and \( u(t) \in L^2(0, T) \) then there exists a unique solution \( \{U, V\}^T \in C([0, T]; X_1 \times X_0) \).
Spectral analysis of $\mathcal{A}$ for distinct wave speeds

Assume $\sqrt{\frac{K}{\alpha}}, \sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}, \ldots \sqrt{\frac{E_n}{\rho_n}}$ are distinct.

Setting $\mathcal{A}Y = \lambda Y$ is equivalent to

$$V = \lambda U, \quad A_1 U + A_2 \lambda U = \lambda^2 U.$$ 

**The case $\lambda = 0$:** We find a null vector of which corresponds to a zero-energy state of uniform lateral translation. Also there is an associated generalized eigenvector corresponding to constant lateral motion.
The case $\lambda \neq 0$: The eigensystem can be written as

\begin{align*}
  m\lambda^2 u - \alpha \lambda^2 D_x^2 u + K D_x^4 u - D_x N^T h_\varepsilon (G + \lambda \tilde{G}) \varphi &= 0 \\
  h_\mathcal{O} p_\mathcal{O} \lambda^2 u - h_\mathcal{O} E_\mathcal{O} D_x^2 u + B^T (G + \lambda \tilde{G}) \varphi &= 0 \\
  Bu + h_\varepsilon ND_x u &= \varphi.
\end{align*}

We look for solutions of the form

\begin{align*}
  u &= \frac{a}{\sigma_k} \sin \sigma_k x, \quad \mathbf{u} = \mathbf{C}_k \cos \sigma_k x; \quad \sigma_k = k \pi / L, \quad \mathbf{C}_k = (c_1, c_3, \ldots, c_n)^T.
\end{align*}

Solutions of this form satisfy all the homogeneous boundary conditions and lead to the following algebraic system of equations for $k = 0, 1, 2, \ldots$:

\begin{align*}
  \frac{a}{\sigma_k} m\lambda^2 + \frac{a}{\sigma_k} \alpha \lambda^2 \sigma_k^2 + \frac{a}{\sigma_k} K \sigma_k^4 + \sigma_k N^T h_\varepsilon (G + \lambda \tilde{G}) (\frac{B}{h_\varepsilon} \mathbf{C} + \frac{a}{\sigma_k} N \sigma_k) &= 0 \\
  h_\mathcal{O} p_\mathcal{O} \lambda^2 \mathbf{C} + h_\mathcal{O} E_\mathcal{O} \sigma_k^2 \mathbf{C} + B^T (G + \lambda \tilde{G}) (\frac{B}{h_\varepsilon} \mathbf{C} + \frac{a}{\sigma_k} N \sigma_k) &= 0.
\end{align*}
For each $k; \ k = 0, 1, 2, \ldots$ there are $2m + 4 = n + 3$ eigenvalues occurring in complex conjugate pairs $\{\lambda_{k,j}^\pm\}; \ j = 0, 1, 3, \ldots n$ where

$$\lambda_{k,0}^\pm = -\frac{N^T \tilde{G} h \varepsilon N}{2\alpha} \pm i\sigma_k \sqrt{\frac{K}{\alpha}} + \mathcal{O}(k^{-1})$$

$$\lambda_{k,j}^\pm = -\tilde{\gamma}_j - 1 + \tilde{\gamma}_{j+1} + 2 \sqrt{\frac{E_j}{\rho_j}} \pm i\sigma_k \sqrt{\frac{E_j}{\rho_j}} + \mathcal{O}(k^{-1}), \ j = 1, 3, \ldots n.$$  

where $\tilde{\gamma}_0 = \tilde{\gamma}_{n+1} = 0$ and for $j = 2, 4, \ldots 2m$ $\tilde{\gamma}_j = \tilde{G}_j / h_j$. 
Associated eigenvectors (and possibly generalized eigenvectors) $Y_\lambda$ are of the form

$$Y_\lambda = \left( \begin{array}{c} \frac{U_\lambda}{\lambda} \\ U_\lambda \end{array} \right); \quad U_\lambda = \left( \begin{array}{c} u_{k,j} \\ u_{k,j} \end{array} \right) = \left( \begin{array}{c} \frac{A_{k,j}}{\sigma_k} \sin(\sigma_k x) \\ \vec{B}_{k,j} \cos(\sigma_k x) \end{array} \right),$$

where for $k$ sufficiently large the $Y_{\lambda_\pm}^{\pm}_{k,j}$ are eigenvectors and

$$\begin{pmatrix} A_{k,0} & A_{k,1} & A_{k,3} & \cdots & A_{k,n} \\ \vec{B}_{k,0} & \vec{B}_{k,1} & \vec{B}_{k,3} & \cdots & \vec{B}_{k,n} \end{pmatrix} = \Delta(k) = I_{m+2} + \mathcal{O}(k^{-1}).$$
**Proposition 1** Assume $\tilde{G}$ is nonsingular and the wave speeds are distinct. Then if $m > 1$ every non-zero eigenvalue has negative real part. If $m = 1$ this remains true if

$$\left\{ \frac{K}{\alpha + \frac{m}{\sigma_k^2}} \right\}_{k=1}^{\infty} \bigcap \left\{ \frac{E_i}{\rho_i} \right\}_{i=1,3,\ldots,n} = \emptyset.$$  

**Remark.** If the wave speeds are the same there exists an infinite family of eigenvalues on the imaginary axis corresponding to longitudinal vibrations. In addition, if $\left\{ \frac{K}{\alpha + \frac{m}{\sigma_k^2}} \right\}_{k=1}^{\infty} \bigcap \left\{ \frac{E_1}{\rho_1}, \frac{E_3}{\rho_3} \right\} \neq \emptyset$, and $m = 1$, it is possible to find appropriate physical parameters so that bending motions are possible that do not dissipate energy.
We also find that:

- The eigenfunctions are block orthogonal; i.e., those corresponding to \( \{ \lambda_{k,j}^\sigma \} \), \( (j = 0, 1, 3, \ldots n; \sigma = +, -) \) are orthogonal, relative to the energy inner product to those with different \( k \) index.

- Eigenfunctions form a Riesz basis for finite energy space.

Consequently,

**Corollary** Solutions to the uncontrolled problem (with \( M(t) = 0, \ g_0(t) = 0 \)) have energy that satisfies

\[ E(t) \leq M e^{-\gamma t} E(0), \]

where \( M \) is independent of initial data and \( -\gamma \) is the supremum of the real parts of the eigenvalues of \( A \). In particular, if the hypothesis of Proposition 1 is satisfied then solutions decay exponentially to a state of constant lateral velocity.
Analysis of control problem

We analyze the moment problem for the case of distinct wave speeds. Let us assume that the initial data given is zero, and determine which states are reachable in time \( T \). We write the terminal state as

\[
Y(T) = \sum_{\lambda \in \sigma(A)} c_\lambda Y_\lambda; \quad c_\lambda = < Y(T), Y_\lambda^* >_e
\]

where \( Y_\lambda^* \) is the eigenvector of \( A^* \) with eigenvalue \( \bar{\lambda} \). The above expansion is valid since the eigenfunctions of \( A^* \) are biorthogonal to those of \( A \).

Furthermore, the sequence \( \{ c_\lambda \}_{\lambda \in \sigma(A)} \) satisfies for some \( C > 0 \)

\[
\frac{1}{C} \| \{ c_\lambda \} \|_{\ell^2} \leq \| Y(T) \|_e \leq C \| \{ c_\lambda \} \|_{\ell^2}
\]

by the Riesz basis property.
We first calculate Fourier coefficients of the control input:

\[
\left\langle \begin{pmatrix} 0 \\ \mathcal{B}\{M, g_{\mathcal{O}}\} \end{pmatrix}, Y^*_\lambda \right\rangle_e = \mathcal{B}\{M, g_{\mathcal{O}}\}, V^*_\lambda \right\rangle = \mathcal{B}\{M, g_{\mathcal{O}}\}, V^*_\lambda \right\rangle = \mu_0, \mu_1, \mu_3, \ldots \mu_n \}
\]

Now assume the controls are a linear function of a single scalar control \( \tilde{u}(t) = u(T-t) \):

\[
\{M, g_{\mathcal{O}}\} = (M(t), g_1(t), g_3(t), \ldots g_n(t)) = (\mu_0, \mu_1, \mu_3, \ldots \mu_n)u(T-t).
\]

The above (with \( \lambda = \lambda_{k,j}^\pm \)) becomes

\[
\left\langle \begin{pmatrix} 0 \\ \mathcal{B}\{M, g_{\mathcal{O}}\} \end{pmatrix}, Y^*_{\lambda_{k,j}^\pm} \right\rangle = b_{\lambda_{k,j}^\pm} \tilde{u}(t) = (\mu_j(-1)^k + \mathcal{O}(k^{-1}))\tilde{u}(t),
\]
The variation of parameters solution can be written

\[ Y(T) = \int_0^T e^{A(T-s)} \left( \begin{array}{c} 0 \\ \mathcal{B}\{M, \mathbf{g}_O\} \end{array} \right) ds = \int_0^T e^{At} \left( \begin{array}{c} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_O\} \end{array} \right) ds \]

where \( \tilde{M}(t) = M(T-t) \), \( \tilde{\mathbf{g}}_O = \mathbf{g}_O(T-t) \).

Hence multiplying the above by the eigenvectors \( Y_\lambda^* \) of \( A^* \) gives

\[ c_\lambda = < Y(T), Y_\lambda^* >_e = < \int_0^T e^{At} \left( \begin{array}{c} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_O\} \end{array} \right), Y_\lambda^* >_e \]

\[ = \int_0^T e^{\lambda t} < \left( \begin{array}{c} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_O\} \end{array} \right), Y_\lambda^* >_e dt = \int_0^T e^{\lambda t} b_\lambda u(t) dt. \]
The control problem is thus reduced to solving the following moment problem:

\[ c_\lambda = \int_0^T e^{\lambda t} b_\lambda u(t) \, dt \quad \forall \lambda \in \sigma(A). \]

This infinite system has the formal solution

\[ u = \sum_{\lambda \in \sigma(A)} \frac{c_\lambda}{b_\lambda} q_\lambda, \]

where \( \{q_\lambda(t)\}_{\lambda \in \sigma(A)} \) is a biorthogonal sequence to \( \{e^{\lambda t}\} \).

Thus necessary and sufficient conditions for solvability are:

1. each \( b_\lambda \) must be nonzero,

2. the sequence \( \{q_\lambda\} \) must exist.

3. for any square-summable sequence \( \{c_\lambda\} \), the series on right should converge in \( L^2(0, T) \).
These conditions will be satisfied provided \( \{e^{\lambda t}\}_{\lambda \in \sigma(A)} \) can be shown to be a Riesz basis on \( L^2(0,T) \) (in which case, \( \{q_\lambda\} \) is also a Riesz basis on \( L^2(0,T) \)) and that \( |b_\lambda| \) be bounded below by some positive number.

**Proposition 2** Assume that the eigenvalues \( \{\lambda_{k,j}^\pm\} \)

\((k = 1, 2, \ldots; j = 0, 1, 3, \ldots n)\) are distinct and satisfy

\[
\lambda_{k,j}^\pm = -a_j \pm i\beta_j k + z_{k,j}^\pm,
\]

where \( \beta_j > 0, (j = 0, 1, 3, \ldots n) \), \( a_0 < a_1 < \ldots < a_n \), \( \{z_{k,j}^\pm\} \in l^2 \). Then \( \{\exp(\lambda_{k,j}^\pm t)\} \) forms a Riesz basis for its closed span in \( L^2(0,T) \), for any \( T > \tau \) where

\[
\tau = \sum_{j=0,1,3,\ldots n} \frac{2\pi}{\beta_j}.
\]
In our particular problem, this condition is satisfied if the damping coefficients \( \tilde{G}_j, j = 2, 4, \ldots 2m \) are chosen so that

\[
-N^T \tilde{G} h \varepsilon N \frac{\tilde{\gamma}_j - \tilde{\gamma}_j + 1}{2\alpha} - \frac{\tilde{\gamma}_j - 1 + \tilde{\gamma}_j + 1}{2 h_j \rho_j}, \quad j = 1, 3, \ldots n
\]

are distinct

where \( \tilde{\gamma}_0 = \tilde{\gamma}_{n+1} = 0 \) and for \( j = 2, 4, \ldots 2m \), \( \tilde{\gamma}_j = \tilde{G}_j / h_j \).

It is easy to find sets of damping parameters \( \tilde{G}_j \) for which this condition holds. Furthermore, it is possible to show that by perturbing \( \tilde{G} \) and \( G \) (if necessary) all the eigenvalues can be forced to be distinct.

**Theorem 2** By picking \( \tilde{G} \) and (if necessary) \( G \) appropriately, the hypothesis of Proposition 2 is satisfied and thus \( \{ \exp(\lambda_{k,j}^\pm t) \} \) forms a Riesz basis for its closed span in \( L^2(0, T) \), for any \( T > \tau^* \) where

\[
\tau^* = \sum_{j=0,1,3,\ldots n} \frac{2L}{\mu_j}; \quad (\mu_j = j^{th} \text{ wave speed})
\]

(1)
Finally, to solve the moment problem, it is required that the control input coefficients $|b_\lambda|$ be bounded away from zero.

Asymptotically, (for large $k$) $|b_{\lambda_{k,j}}| = \mu_j$. Thus it is necessary that $\mu_j \neq 0$ for $j = 0, 1, 3, \ldots n$. If this is satisfied, there are at most finitely many, say $N$ modes to check. Controllability corresponding to the $k$th modal control system is easily checked by the Kalman rank condition.

It is possible to show that there exists many choices of control law vector $\mu = (\mu_0, \mu_1, \mu_3, \ldots \mu_n)$ which results in a controllable system.

We summarize these in the following theorem:
Theorem 3. Suppose the wave speeds are distinct and

$$(M(t), g_0(t)) = \mu u(t) = (\mu_0, \mu_1, \mu_3, \ldots \mu_n)u(t)$$

where $\mu_i \neq 0$ for $i = 0, 1, 3, \ldots n$. For appropriate choice of $G_j$, $\tilde{G}_j$, $j = 2, 4, \ldots 2m$, all nonzero eigenvalues of $A$ are distinct. In this case, if the $\mu_j$’s, $(j = 0, 1, 3, \ldots n)$ are picked so that all $b_{\lambda}$ are nonzero, then any initial state can be controlled to a state of uniform lateral motion in time $T > \tau^*$. 
Comments

1. In the case of where two or more wave speeds are the same, it is possible to have two distinct branches of eigenvalues that are both asymptotically undamped. In this case, there is no uniform gap between eigenvalues (regardless of choice of $G$, $\tilde{G}$). In this case it is necessary to use two or more controls to obtain a controllable system.

2. If full boundary control is used, exact controllability holds in time

$$T = \max \left\{ \frac{2L}{\mu_j} \right\} \quad (\mu_j = j\text{th wave speed}).$$

3. Same idea of “tuning damping parameters” applies to a variety of coupled hyperbolic systems with light damping.