Distributed control of a cochlea model *

Isaac Chepkwony[†], *Scott W. Hansen*[‡]

1 Introduction

The cochlea is the sensory organ inside the inner ear that converts sound to nerve impulses that respond to specific frequencies. There is a long and fascinating history of mathematical modeling of the cochlea going back to Hemholtz (1863) [6], who suggested that the cochlea contains a sequence of resonators that each resonate at a specific frequency. This theory survived for many years until experiments by Békésy [2] in the 1930's and 1940's indicated travelling wave phenomena in the basilar membrane motions. This contradicted the Hemholz theory and indicated the essential nature of the fluid-elastic interaction in explaining the mechanics of the cochlea. Since the 1950's many mathematical models for the cochlea have been proposed which incorporate the fluid-elastic coupling of the cochlear fluid and the basilar membrane. It was soon discovered that the mathematical models do not predict the sharp level of tuning seen in *en-vivo* cochlea, and hence an active tuning mechanism was suggested. Since then much of the modeling work has focused on describing the mechanics of this active internal feedback. To date there remains much debate about the precise nature of this internal feedback. For more on the history of cochlea modeling see Luce [7].

The goal of this paper is to begin an analysis of the controllability structure of a cochlea. Roughly, we wish to determine the extent to which controlling the displacements over a small portion of the cochlea (part of the basilar membrane or oval window, for example) for a short period of time influences the motion of the entire basilar membrane. A better understanding of the controllability structure of a cochlea could be helpful in the design of hearing aids and cochlear implants.

Since this is the first discussion of controllability of a cochlea model that we are aware of, we consider a very basic *passive* cochlea model, i.e., without the complex

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*Second author supported in part by the National Science Foundation under grant DMS02052148.

[†]Department of Mathematics, Kenyatta University, Nairobi, Kenya

[‡]Department of Mathematics, Iowa State University, Ames, Iowa, 50011

effects of active internal feedback. The cochlear fluid will be assumed to be a two-dimensional linear potential fluid. Flexible portions of the fluid boundary (i.e, the basilar membrane, the oval window and round window) are modeled as continuous arrays of springs. Such assumptions are generally well-accepted, (see Neely, [10]) at least as a first approximation and are used in many papers, e.g., Xin [12], Neely [10], Allen [1].

There are two main results of this article. First we prove well-posedness of the the cochlea model in appropriate function spaces. Secondly, we prove that approximate controllability holds with control active on an arbitrarily small open subset of the basilar membrane. For purposes of simple exposition, we initially assume a "rectangular geometry" of the cochlea assumed in many papers and a simple basilar membrane model that can be viewed as an array of independent springs coupled only through the fluid. We comment in the end on more general geometries and other basilar membrane models for which the same results hold.

We mention that while there exists a fairly large volume of literature on control of fluid-elastic structures, there seems to be very few articles relevant to controllability of the cochlea model due to the specific geometry of the cochlea and the high variability of elastic moduli. Here we mention only a few of the references that we are aware of that are most relevant to controllability of a cochlea. Hansen and Lyashenko [5], Hansen [4] proved exact controllability results for beam and membrane fluid elastic systems with constant coefficient elastic moduli. Chepkwony [3] proved a slightly weaker version on our main result in his thesis, and also used a multiplier method to establish controllability of a cochlea with a small longitudinal membrane elastic coupling, but with limited variability of basilar membrane stiffnesses.

This paper is organized as follows: In section 2 we formulate the cochlea model in the rectangular geometry described by diagram Figure 1. In section 3 we prove well-posedness of the model. In section 4 we prove the main controllability results and comment on a number of generalizations.

2 Cochlea model

For the purpose of explaining the model, we consider the two-dimensional polygonal geometry for the cochlear cavity indicated in Figure 1. (Our results remain valid for more general Lipschitz domains; see Remark 1.) The cochlear cavity consists of the domain of the fluid Ω together with its fixed boundary Γ_0 and flexible boundary $\Gamma_{OW} \cup \Gamma_{RW} \cup \Gamma_{BM}$, representing the oval window (OW), round window (RW) and basilar membrane (BM). The BM itself the union of the upper and lower portions of the BM: $\Gamma_{BM} = \Gamma^+_{BM} \cup \Gamma^-_{BM}$.

The BM displacement (in the y-direction) from equilibrium is given by $w(x,t)$, $0 < x < L_{BM},\ t > 0$. The displacement (in the x direction) from equilibrium of the OW and RW are described by $\eta(y,t)$ on Γ_{OW}, $t > 0$ and $\xi(y,t)$ on Γ_{RW}, $t > 0$. respectively.

The vibrations of the membranes on the oval and round windows are modeled as spring-mass systems. For simplicity we assume the displacements are constant with respect to y over their respective domains, i.e., $\eta(y,t) = \eta(t)$ and $\xi(y,t) = \xi(t)$. (Any one parameter family of motions would lead to essentially the same model.)

The fluid is assumed to have a constant density ρ and a velocity potential satisfying Laplace's equation. That is, the fluid velocity is given by $\nabla\psi$, where ψ is a solution of $\Delta\psi = 0$ on Ω. Consequently, the velocity of the fluid must match the velocity of moving parts of the boundary. Thus

$$\left.\begin{array}{lllllll} \Delta\psi & = & 0 & \text{in} & \Omega & & \\ \psi_n & = & 0 & \text{on} & \Gamma_0 & & \\ \psi_n & = & -\psi_y & = & -w_t & \text{on} & \Gamma_{BM}^+ \\ \psi_n & = & \psi_y & = & w_t & \text{on} & \Gamma_{BM}^- \\ \psi_n & = & -\psi_x & = & \eta_t & \text{on} & \Gamma_{OW} \\ \psi_n & = & -\psi_x & = & \xi_t & \text{on} & \Gamma_{RW} \end{array}\right\}, \tag{1}$$

In (1), Ω is assumed to be a fixed domain so that a linear problem is obtained. Hence the velocities are matched on the equilibrium positions of the flexible boundary.

In addition to the displacement assumptions, the motions of the oval and round windows are constrained by the incompressibility of the fluid. Hence

$$\xi = -(L_{OW}/L_{RW})\eta. \tag{2}$$

The incompressibility constraint does not however restrict the possible motions of the BM since a displacement in the upper chamber is balanced by an opposite displacement in the lower chamber.

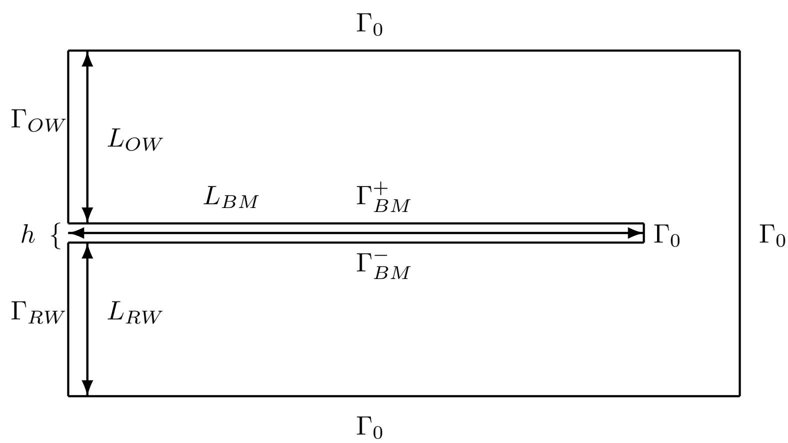

Figure 1. The domain of the cochlear fluid.

The BM is modeled as a continuous array of springs with linear mass density $m_0(x)$; $0 < x < L_{BM}$, stiffness density $k_0(x)$; $0 < x < L_{BM}$. Typically, the stiffness decreases by several orders of magnitude over the length of the cochlea, hence nonconstant stiffness is essential to the model.

The spring constants for the oval windows and round windows are (for simplicity) described by constant mass densities $m_1 = m_1(y)$; $(x, y) \in \Gamma_{OW}$, $m_2 = m_2(y)$; $(x, y) \in \Gamma_{RW}$ and constant stiffness densities $k_1 = k_1(y)$ $(x, y) \in \Gamma_{OW}$, $k_2 = k_2(y)$ $(x, y) \in \Gamma_{RW}$.

The energy $\mathcal{E}(t)$ is the sum of the kinetic $K(t)$ and potential $P(t)$ energies where

$$K = \frac{1}{2}\int_{\Omega} \rho|\nabla\psi|^2 \, d\Omega + \frac{1}{2}\int_0^{L_{BM}} m_0 w_t^2 \, dx + \frac{1}{2}\int_{\Gamma_{OW}} m_1 \eta_t^2 \, dy + \frac{1}{2}\int_{\Gamma_{RW}} m_2 \xi_t^2 \, dy$$

and

$$P = \frac{1}{2}\int_0^{L_{BM}} k_0 w^2 dx + \frac{1}{2}\int_{\Gamma_{OW}} k_1 \eta^2 dy + \frac{1}{2}\int_{\Gamma_{RW}} k_2 \xi^2 dy.$$

We assume the BM is subject to external applied force density $F_0(x, t)$, $0 < x < L_{BM}$, while the oval window is subject to external applied constant force density $F_1(t)$ on Γ_{OW}. The work integral is given by

$$W(t) = \int_0^{L_{BM}} F_0 w dx + \int_0^{L_{OW}} F_1 \eta dy.$$

The equations of motion can be obtained from Hamilton's principle. That is, the first variation, with respect to a class of admissible variations, of the lagrangian $\mathcal{L} = \int_0^T (K + W - P) dt$ is set to zero.

Using the displacement assumptions and constraint (2) the Lagrangian can be written

$$\begin{aligned}\mathcal{L} = \int_0^T \{ & \frac{1}{2}\int_{\Omega} \rho|\nabla\psi|^2 \, d\Omega + \frac{1}{2}\int_0^{L_{BM}} (m_0(x){w_t}^2 - k_0(x) w^2) \, dx \\ & + \frac{L_{OW}}{2}(m_1 {\eta_t}^2 - k_1 \eta^2) + \frac{L_{OW}^2}{L_{RW}}(m_2 \eta_2^2 - k_2 \eta^2) + \int_0^{L_{BM}} F_0 w \, dx + L_{OW} F_1 \eta\end{aligned}$$

From Hamilton's principle the equations of motion take the form:

$$m_0(x) w_{tt} + k_0(x) w + [\rho\psi_t]_{\Gamma_{BM}} = F_0, \quad x \in (0, L_{BM}), \, t > 0 \quad (3)$$

$$(\operatorname{Avg}_{\Gamma_{OW}}(\rho\psi_t) - \operatorname{Avg}_{\Gamma_{RW}}(\rho\psi_t)) + \hat{m_1}\eta_{tt} + \hat{k_1}\eta = F_1, \quad t > 0 \quad (4)$$

where ψ satisfies (1) and we have used the notation

$$\begin{aligned}[\rho\psi_t]_{\Gamma_{BM}} &= \rho\psi_t|_{\Gamma_{BM}^-} - \rho\psi_t|_{\Gamma_{BM}^+} \\ \hat{m_1} &= (m_1 + \frac{L_{OW}}{L_{RW}} m_2) \\ \hat{k_1} &= (k_1 + \frac{L_{OW}}{L_{RW}} k_2) \\ \operatorname{Avg}_{\Gamma_{OW}}(\rho\psi_t) - \operatorname{Avg}_{\Gamma_{RW}}(\rho\psi_t) &= \frac{1}{L_{OW}}\int_0^{L_{OW}} (\rho\psi_t) dy - \frac{1}{L_{RW}}\int_0^{L_{RW}} (\rho\psi_t) dy.\end{aligned}$$

and where ψ satisfyies

$$\left.\begin{array}{l}\Delta\psi = 0 \text{ in } \ \Omega \\ \psi_n = 0 \text{ on } \Gamma_0 \\ \psi_n = -w_t \text{ on } \Gamma^+_{BM} \\ \psi_n = w_t \text{ on } \Gamma^-_{BM} \\ \psi_n = \eta_t \text{ on } \Gamma_{OW} \\ \psi_n = -\frac{L_{OW}}{L_{RW}}\eta_t \text{ on } \Gamma_{RW}\end{array}\right\}, \tag{5}$$

The natural energy space E for the system is

$$E = (w, \eta, w_t, \eta_t, \psi) \in L^2(\Gamma_{BM}) \times \mathbf{R} \times L^2(\Gamma_{BM}) \times \mathbf{R} \times H^1(\Omega).$$

Since the energy of the fluid is completely determined by the Neumann data on Γ, the fluid can be eliminated as a state variable. To this end we define the Neumann to Dirichlet map, Λ_0 as follows:

$$y = \Lambda_0 f \quad \Longleftrightarrow \quad \left\{\begin{array}{lcl}\Delta\psi & = & 0 \text{ on } \Omega \\ \psi_n & = & f \text{ on } \Gamma. \\ y & = & \psi\big|_\Gamma \ \text{ on } \Gamma.\end{array}\right.$$

Then $\Lambda_0 : \tilde{H}^{-1/2}(\Gamma) = \{z \in H^{-1/2}(\Gamma) :< z, 1 >= 0\} \to H^{1/2}(\Gamma) + C$, where $H^{1/2}(\Gamma) + C$ denotes the quotient space of $H^{1/2}(\Gamma)$ functions identified up to an additive constant.

Since Ω is a Lipschitz domain, it is well-known (see [8]) that

$$\Lambda_0 : \tilde{L}^2(\Gamma) \to H^1(\Gamma) + C \qquad \text{continuously.} \tag{6}$$

(Here and later we use $\tilde{M}$ to indicate the subspace of M orthogonal to constants and $M + C$ for the quotient space where elements of M are identified if they differ by a constant.) Furthermore, it is shown in [5] that Λ_0 is self-adjoint in the sense that

$$\int_\Gamma g\Lambda_0 f \, d\Gamma = \int_\Gamma f\Lambda_0 g \, d\Gamma \qquad \forall f, g \in \tilde{L}^2(\Gamma). \tag{7}$$

For $g \in L^2(0, L_{BM})$, $\alpha \in \mathbb{R}$ define

$$\tilde{g} = \left\{\begin{array}{ll}0 & \text{on } \Gamma_0 \cup \Gamma_{OW} \cup \Gamma_{RW} \\ -g & \text{on } \Gamma^+_{BM} \\ g & \text{on } \Gamma^-_{BM}\end{array}\right., \qquad \tilde{\alpha} = \left\{\begin{array}{ll}0 & \text{on } \Gamma_0 \cup \Gamma^+_{BM} \cup \Gamma^-_{BM} \\ \alpha & \text{on } \Gamma_{OW} \\ -\frac{L_{OW}}{L_{RW}}\alpha & \text{on } \Gamma_{RW}\end{array}\right. \tag{8}$$

and define

$$\begin{array}{ll}\Lambda : L^2(0, L_{BM}) \to H^1(0, L_{BM}) : & \Lambda g = \Lambda_0\tilde{g}\,|_{\Gamma^-_{BM}} - \Lambda_0\tilde{g}\,|_{\Gamma^+_{BM}} \\ \mathrm{S} : \mathbb{R} \to H^1(0, L_{BM}) : & \mathrm{S}\alpha = (\Lambda_0\tilde{\alpha}\,|_{\Gamma^-_{BM}} - \Lambda_0\tilde{\alpha}\,|_{\Gamma^+_{BM}}) \\ \mathrm{T} : L^2(0, L) \to \mathbb{R} : & \mathrm{T}g = L_{OW}(\ \mathrm{Avg}_{\Gamma_{OW}}\Lambda_0\tilde{g}|_{\Gamma_{OW}} - \ \mathrm{Avg}_{\Gamma_{RW}}\Lambda_0\tilde{g}|_{\Gamma_{RW}}) \\ \mathrm{H} : \mathbb{R} \to \mathbb{R} : & \mathrm{H}\alpha = L_{OW}(\mathrm{Avg}_{\Gamma_{OW}}\Lambda_0\tilde{\alpha} - \ \mathrm{Avg}_{\Gamma_{RW}}\Lambda_0\tilde{\alpha}).\end{array}$$

The cochlea system (3), (4), (5) can now be written:

$$\begin{array}{ll} (m_0 + \rho\Lambda)w_{tt} + \rho\mathrm{S}\eta_{tt} + k_0 w = F_0 & \text{on } (0, L_{BM}) \times \mathbb{R}^+ \\ (\hat{m}_1 + \rho\mathrm{H})\eta_{tt} + \rho\mathrm{T}w_{tt} + \hat{k}_1\eta = F_1 & \text{on } \mathbb{R}^+. \end{array} \tag{9}$$

We consider initial conditions of the form

$$(w, \eta, w_t, \eta_t)|_{t=0} = (w^0, \eta^0, w^1, \eta^1). \tag{10}$$

3 Existence and Uniqueness of Solutions

Proposition 1. $R = \begin{pmatrix} \Lambda & S \\ T & H \end{pmatrix}$ *is positive, compact and self-adjoint on* $\mathbf{X} = L^2(0, L_{BM}) \times \mathbb{R}$*. Furthermore* $R : \mathbf{X} \to H^1(0, L_{BM}) \times \mathbb{R}$ *continuously.*

Proof: From the definition of Λ and (6) we have

$$\begin{aligned} \| \Lambda g \|_{H^1(0,L_{BM})} &= \| \Lambda_0 \tilde{g} \mid_{\Gamma^-_{BM}} - \Lambda_0 \tilde{g} \mid_{\Gamma^+_{BM}} \|_{H^1(0,L_{BM})} \\ &\leq \| \Lambda_0 \tilde{g} \|_{H^1(0,L_{BM})} + \| \Lambda_0 \tilde{g} \|_{H^1(0,L_{BM})} \\ &\leq C \| \tilde{g} \|_{\tilde{L}^2(0,L_{BM})} = 2C \|g\|_{L^2(0,L_{BM})}. \end{aligned}$$

Thus Γ has the required continuity property. Similar arguments establish the required continuity estimates for operators S, T, H. The compactness of R on $\mathbf{X}$ follows.

To prove self-adjointness, we show that Λ and H are self-adjoint and $\mathrm{S} = \mathrm{T}^*$. If $f, g \in L^2(0, L_{BM})$,

$$\begin{aligned} < \Lambda f, g > &= \int_\Gamma (\Lambda_0 \tilde{f} \mid_{\Gamma^-_{BM}} - \Lambda_0 \tilde{f} \mid_{\Gamma^+_{BM}}) g \, dx = \int_{\Gamma^-_{BM}} (\Lambda_0 \tilde{f}) \tilde{g} \, dx + \int_{\Gamma^+_{BM}} (\Lambda_0 \tilde{f}) \tilde{g} \, dx \\ &= \int_\Gamma \tilde{g} \Lambda_0 \tilde{f} \, d\Gamma = \int_\Gamma \tilde{f} \Lambda_0 \tilde{g} \, d\Gamma \qquad \text{(using (7))} \\ &= \int_{\Gamma^-_{BM}} f \Lambda_0 \tilde{g} \, dx + \int_{\Gamma^+_{BM}} -f \Lambda_0 \tilde{g} \, dx = \int_\Gamma f (\Lambda_0 \tilde{g} \mid_{\Gamma^-_{BM}} - \Lambda_0 \tilde{g} \mid_{\Gamma^+_{BM}}) \, dx \\ &= < f, \Lambda g > . \end{aligned}$$

Thus Λ is self-adjoint.

The previous calcuation also shows that for $f \in L^2(0, L_{BM})$,

$$< \Lambda f, f > = \int_\Gamma \tilde{f} \Lambda_0 \tilde{f} \, d\Gamma. \tag{11}$$

Let ψ be a solution of

$$\Delta \psi = 0 \qquad \text{on } \Omega, \qquad \psi_n = \tilde{f} \qquad \text{on } \Gamma. \tag{12}$$

Then $\Lambda_0 \tilde{f} = \psi|_\Gamma$. Thus use of the divergence theorem gives

$$\int_\Gamma \tilde{f} \Lambda_0 \tilde{f} \, d\Gamma = \int_\Gamma (\psi_n|_\Gamma)(\psi|_\Gamma) \, d\Gamma = \int_\Omega |\nabla \psi|^2 \, d\Omega$$

This combined with (11) shows that Λ is nonnegative. However, if $\nabla\psi$ vanishes on Ω then $\tilde{f}$ vanishes on Γ and hence $f = 0$. Thus Λ is positive.

Next consider

$$\begin{array}{rcl} \Delta\psi &=& 0 \text{ on } \Omega \\ \psi_n &=& \tilde{\eta} \text{ on } \Gamma, \end{array} \qquad \text{where } \tilde{\eta} = \begin{cases} 0 & \text{on } \Gamma_0 \cup \Gamma_{BM}^+ \cup \Gamma_{BM}^- \\ \eta & \text{on } \Gamma_{OW} \\ -\frac{L_{OW}}{L_{RW}}\eta & \text{on } \Gamma_{RW} \end{cases}, \tag{13}$$

where η is a constant function.

Recall $\mathrm{H}\eta = L_{OW}(\mathrm{Avg}_{\Gamma_{OW}}\Lambda_0\tilde{\eta} - \mathrm{Avg}_{\Gamma_{RW}}\Lambda_0\tilde{\eta})$. We first note that

$$\begin{aligned} < \mathrm{H}\eta, v > &= < L_{OW}(\mathrm{Avg}_{\Gamma_{OW}}\Lambda_0\tilde{\eta} - \mathrm{Avg}_{\Gamma_{RW}}\Lambda_0\tilde{\eta}), v > \\ &= L_{OW}((\frac{1}{L_{OW}}\int_{\Gamma_1}\Lambda_0\tilde{\eta})v - (\frac{1}{L_{RW}}\int_{\Gamma_{RW}}\Lambda_0\tilde{\eta})v) \\ &= L_{OW}(\frac{1}{L_{OW}}\int_{\Gamma_{OW}} v\mid_{\Gamma_{OW}}\Lambda_0\tilde{\eta} - \frac{1}{L_{RW}}\int_{\Gamma_{RW}} v\mid_{\Gamma_{RW}}\Lambda_0\tilde{\eta}) \\ &= L_{OW}(\frac{1}{L_{OW}}\int_{\Gamma_{OW}}\tilde{v}\Lambda_0\tilde{\eta} - \frac{1}{L_{RW}}\int_{\Gamma_{RW}} -\frac{L_{RW}}{L_{OW}}\tilde{v}\Lambda_0\tilde{\eta}) \\ &= \int_\Gamma \tilde{v}\Lambda_0\tilde{\eta}. \end{aligned} \tag{14}$$

where $\tilde{v}$ is defined in the same way as $\tilde{\eta}$. Now self-adjointness and positivity follows by the same reasoning used to prove these properties for Λ.

Next, recall $\mathrm{S} : \mathbb{R} \to H^1(0, L_{BM})$ is defined by $\mathrm{S}\eta = (\Lambda_0\tilde{\eta}\mid_{\Gamma_{BM}^-} - \Lambda_0\tilde{\eta}\mid_{\Gamma_{BM}^+})$ where $\tilde{\eta}$ is defined by (13) and $\mathrm{T} : L^2(0, L_{BM}) \to \mathbb{R}$ by

$$Tg = L_{OW}(\mathrm{Avg}_{\Gamma_{OW}}(\Lambda_0\tilde{g}|_{\Gamma_{OW}}) - \mathrm{Avg}_{\Gamma_{RW}}(\Lambda_0\tilde{g}|_{\Gamma_{RW}})),$$

where $\tilde{g}$ is defined from g by (8). Calculations similar to the one leading to (14) reveals that

$$< g, \mathrm{S}\eta > = \int_\Gamma \tilde{g}\Lambda_0\tilde{\eta}$$

and

$$< \mathrm{T}g, \eta > = \int_\Gamma \tilde{\eta}\Lambda_0\tilde{g}.$$

Hence using (7) it follows that $\mathrm{T}^* = \mathrm{S}$. This completes the proof.

In matrix form the initial value problem (9), (10) is:

$$MV_{tt} + \rho RV_{tt} + KV = F, \qquad V(0) = V^0,\ V'(0) = V^1 \tag{15}$$

where

$$M = \begin{pmatrix} m_0 & 0 \\ 0 & \tilde{m_1} \end{pmatrix}, R = \begin{pmatrix} \Lambda & \mathrm{S} \\ \mathrm{T} & \mathrm{H} \end{pmatrix}, K = \begin{pmatrix} k_0 & 0 \\ 0 & \tilde{k_1} \end{pmatrix}, V = \begin{pmatrix} w \\ \eta \end{pmatrix},$$

$$F = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}, \quad V^0 = \begin{pmatrix} w^0 \\ \eta^0 \end{pmatrix}, \qquad V^1 = \begin{pmatrix} w^1 \\ \eta^1 \end{pmatrix}.$$

Define the energy inner product e on $\mathbf{X}^2 = (L_2(\Gamma_{BM}) \times \mathbf{R})^2$ by

$$e\left(\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}\right) = a\left(\left(U_1 \right), \left(V_1 \right)\right) + b\left(\left(U_2 \right), \left(V_2 \right)\right)$$

where

$$a(V, \hat{V}) = (KV, \hat{V})_{\mathbf{X}\times\mathbf{X}}, \quad b(V, \hat{V}) = ((M + \rho R)V, \hat{V})_{\mathbf{X}\times\mathbf{X}}, \quad \forall V, \hat{V} \in \mathbf{X}.$$

We assume that the mass distribution m_0 and spring stiffness distribution k_0 are strictly positive and bounded on $[0, L_{BM}]$. It follows from Proposition 1 that

$$\tilde{A} = (M + \rho R)^{-1} K$$

is strictly positive, bounded and self-adjoint relative to $b(.,.)$. In first order form the system (15) becomes:

$$v_t = \mathcal{A}v + \mathcal{F}, \qquad v(0) = v^0 = \begin{pmatrix} V^0 \\ V^1 \end{pmatrix}. \tag{16}$$

where

$$v = \begin{pmatrix} V \\ V_t \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & I \\ -\tilde{A} & 0 \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 \\ (M + \rho R)^{-1} F \end{pmatrix}.$$

It can easily be shown that $\mathcal{A}$ is skew-adjoint with respect to the energy inner product e. Hence by Stone's theorem, (e.g., see [11]) $\mathcal{A}$ is the generator of a unitary group. Moreover boundedness of the generator implies that the group is uniformly continuous. Thus we have the following:

Theorem 2. *Assume that m_0 and k_0 are bounded and strictly positive on $[0, L_{BM}]$. Then $\mathcal{A}$ is an infinitesimal generator of a uniformly continuous group of unitary operators on $\mathbf{X}^2 = (L_2(\Gamma_{BM}) \times \mathbb{R})^2$. Given the control $F \in L_2(0, \tau; X)$, V^0, V^1 $\in \mathbf{X}$ the solution V of initial value problem*

$$(M + \rho R)V_{tt} + KV = F \quad \textit{with} \quad V(0) = V^0, \quad V_t(0) = V^1,$$

satisfies

$$V \in C([0, \tau]; \mathbf{X}).$$

Furthermore if $F \in C^k(\mathbb{R}; \mathbf{X})$ then $V \in C^{k+2}(\mathbb{R}; \mathbf{X})$.

4 Approximate Controllability

Theorem 3. *Let ω be an open subset of $(0, L_{BM})$. Given V^0, V^1, W^0, $W^1 \in \mathbf{X}$, $T > 0$, and $\epsilon > 0$, there exists $F_0 \in L^2((0, T) \times \omega)$ for which the solution at time T of (15) with $F_1 \equiv 0$ satisfies*

$$\|(V(T), V_t(T)) - (W^0, W^1)\|_{\mathbf{X}^2} < \epsilon.$$

Proof: Since the semigroup is unitary, we may assume V^0 and V^1 are zero and prove the that the map $\Phi_T : L^2(\omega \times (0,T)) \to \mathbf{X}^2$ that takes the control F_0 to the solution at time T has dense range. Equivalently we need to show Φ_T^* is one to one. Putting $\Phi_T^* Z = 0$ is equivalent to the following "observed problem":

$$m_0 w_{tt} + \rho\Lambda w_{tt} + \rho \mathrm{S}\eta_{tt} + k_0 w = 0 \quad \text{on } (0, L_{BM}) \times [0,T] \tag{17}$$

$$\hat{m}_1 \eta_{tt} + \rho \mathrm{T} w_{tt} + \rho \mathrm{H}\eta_{tt} + \hat{k}_1 \eta = 0 \quad \text{on } [0,T] \tag{18}$$

$$w_t = 0 \quad \text{on } \omega \times [0,T] \tag{19}$$

$$(w, \eta, w_t, \eta_t) = Z = (z_1^0, z_2^0, z_1^1, z_2^1) \quad t = 0 \tag{20}$$

we wish to show that $Z = 0$. Since by Theorem 2, solutions are C^∞ in time, the equations may be differentiated in time. Hence we have $w_{tt} = 0$ on ω. It unfortunately does not (immediately) follow that Tw_{tt} or Sw_{tt} vanishes anywhere due to nonlocal effect of the fluid. By differentiating (17), (19) we obtain

$$\rho\Lambda w_{ttt} + \rho \mathrm{S}\eta_{ttt} = 0 \quad \text{on} \quad \omega \times [0,T] \tag{21}$$

$$\hat{m}_1 \eta_{ttt} + \rho \mathrm{T} w_{ttt} + \rho \mathrm{H}\eta_{ttt} + \hat{k}_1 \eta_t = 0 \quad \text{on } [0,T].$$

Recall that $\Lambda f = y$ means

$$\left.\begin{array}{l} \Delta\psi = 0 \quad \text{in } \Omega \\ \psi|_{\Gamma_{BM}^+} - \psi|_{\Gamma_{BM}^-} = y \\ \partial_n \psi = \tilde{f} = \begin{cases} f & \text{on} \quad \Gamma_{BM}^+ \\ -f & \text{on} \quad \Gamma_{BM}^- \end{cases} \\ \partial_n\psi = 0 \quad \text{on} \quad \Gamma_{OW} \cup \Gamma_0 \cup \Gamma_{RW} \end{array}\right\}. \tag{22}$$

Also, $\mathrm{S}\alpha = g$ means

$$\left.\begin{array}{l} \Delta\psi = 0 \quad \text{in } \Omega \\ \psi|_{\Gamma_{BM}^+} - \psi|_{\Gamma_{BM}^-} = g \\ \partial_n \psi = \tilde{\alpha} \quad \text{on } \Gamma_{OW} \cup \Gamma_{RW} \\ \partial_n\psi = 0 \quad \text{on } \Gamma \backslash \{\Gamma_{OW} \cup \Gamma_{RW}\} \\ \tilde{\alpha} = \begin{cases} \alpha & \text{on} \quad \Gamma_{OW}^+ \\ -\frac{L_{OW}}{L_{RW}}\alpha & \text{on} \quad \Gamma_{RW}^- \\ 0 & \text{otherwise} \end{cases} \end{array}\right\}. \tag{23}$$

By superposition, we have that $z = \Lambda f + \mathrm{S}\alpha$ means

$$\left.\begin{array}{l} \Delta\Theta = 0 \quad \text{in } \Omega \\ \Theta|_{\Gamma_{BM}^+} - \Theta|_{\Gamma_{BM}^-} = z \\ \partial_n \Theta = \begin{cases} \tilde{\alpha} & \text{on} \quad \Gamma_{OW} \cup \Gamma_{RW} \\ \tilde{f} & \text{on} \quad \Gamma_{BM}^- \cup \Gamma_{BM}^+ \\ 0 & \text{on } \Gamma_0 \end{cases} \end{array}\right\}. \tag{24}$$

Thus equation (21) implies (since $w_{ttt} = 0$ on ω)

$$\left.\begin{array}{l} \Delta\Theta = 0 \quad \text{in } \Omega \\ \Theta|_{\omega\cap(\Gamma_{BM}^+)} - \Theta|_{\omega\cap(\Gamma_{BM}^-)} = 0 \\ \partial_n\Theta = 0 \quad \text{on} \quad \omega \end{array}\right\}.$$

The upper chamber of the cochlea (see Fig. 1) is described by cartesian coordinates (x, y) with $y > h/2\}$ and $y < -h/2$ for the lower chamber. Let $\mathcal{D}$ denote the open rectangle $0 < x < L_{BM}$, $h/2 < y < \min(h/2 + L_{OW}, h/2 + L_{RW})$. For $(x, y) \in \mathcal{D}$ define

$$\phi(x, y) = \frac{\Theta(x, y) - \Theta(x, -y)}{2}.$$

We have

$$\left.\begin{array}{l} \Delta\phi = 0 \quad \text{in} \quad \mathcal{D} \\ \phi = 0 \quad \text{on} \quad \omega \\ \frac{\partial\Theta(x,y)}{\partial y} = \frac{\Theta_y(x,\frac{h}{2})+\Theta_y(x,\frac{-h}{2})}{2} = 0 \quad \text{on} \quad \omega \end{array}\right\}. \tag{25}$$

The problem is overdetermined. Thus $\phi \equiv 0$ in $\mathcal{D}$. Thus Θ in (24) is even with respect to y. It follows (using $f = w_{ttt}$ in (22), (24)) that w_{ttt} vanishes on $(0, L_{BM}) \times (0, T)$. Likewise, η_{ttt} must vanish for Θ to be even. Using this information, the differentiated dual system system (21), reduces to

$$\begin{array}{l} k_0 w_t = 0 \quad \text{on} \quad (0, L_{BM}) \times [0, T] \\ \hat{k_1}\eta_t = 0 \qquad \text{on} \quad [0, T]. \end{array}$$

Under our positivity assumptions for k_0 $\hat{k}_1$, we obtain

$$\begin{array}{l} w_t = 0 \quad \text{on} \quad (0, L_{BM}) \times [0, T] \\ \eta_t = 0 \qquad \text{on} \quad [0, T]. \end{array}$$

This together with (17), (18) result in

$$\begin{array}{l} w = 0 \quad \text{on} \quad (0, L_{BM}) \times [0, T] \\ \eta = 0 \qquad \text{on} \quad [0, T]. \end{array}$$

Thus $Z = 0$ as required. This completes the proof.

Remark 1. The basilar membrane was assumed to have a small thickness $h > 0$ so that Ω is a Lipschitz domain. If $h = 0$ the regularity of Proposition 1 will not hold. Symmetry of a portion of the domain adjacent to the BM, RW, OW with respect to the BM was also used in proving controllability. Otherwise, the rectangular shape taken for the cochlea is inessential and the contollability result remains valid for more general domains subject to the Lipschitz condition and the symmetry condition.

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